AMERICAN MATHEMATICAL MONTHLY

THE NOVEMBER MEETING OF THE IOWA SECTION.

The ninth regular meeting of the Iowa Section was held in the Chamber of Commerce library, Des Moines, on November 4, 1921, the Section, as is usual at the fall meeting, uniting with the Iowa Association of Mathematics Teachers. The vice-chairman, Professor C. W. EMMONS, presided.

One hundred and one persons were present, including the following thirteen members of the Association:

O. W. Albert, Julia T. Colpitts, I. S. Condit, C. W. Emmons, R. B. McClenon, F. M. McGaw, J. V. McKelvey, E. A. Pattengill, J. F. Reilly, H. L. Rietz, Maria M. Roberts, B. F. Simonson, E. R. Smith.

The portion of the program given by the Iowa Section consisted of a report on "The work of the National Committee on Mathematical Requirements" by Professor J. V. MCKELVEY, Iowa State College. Speaking as chairman of the Section's committee, of which Professors Emmons and Albert were the other members, Professor McKelvey discussed the suggestions and recommendations of the National Committee. His paper covered the general scope of the National Committee's report, lists of subjects to be included and those to be excluded, with some special mention of the functional concept in elementary mathematics, and of college entrance examinations.

Professors Rietz and Condit participated in the discussion.

J. F. REILLY, *Secretary.*

THE NOVEMBER MEETING OF THE MISSOURI SECTION.

The fifth regular meeting of the Missouri Section was held in St. Louis, on Friday and Saturday afternoons, November 25–26. The Friday session, held at Soldan High School, was presided over by Professor Louis Ingold, chairman of the Section. The Saturday session, held at Washington University, was a joint meeting with the Southwestern Section of the American Mathematical Society, and was presided over by Professor E. R. Hedrick, chairman of the Southwestern Section, and by Professor Ingold.

There were thirty-three in attendance, including the following twenty-eight members of the Association:

C. Ammerman, C. H. Ashton, A. Davis, O. Dunkel, W. W. Hart, E. R. Hedrick, Louise H. Huff, A. H. Huntington, L. Ingold, S. Lefschetz, W. A. Luby, E. B. Lytle, J. V. McKelvey, U. G. Mitchell, A. R. Nauer, P. R. Rider, P. Robertson, W. H. Roever, I. Roman, W. J. Ryan, A. J. Schwartz, H. E. Slaught,

E. R. Smith, E. Stephens, E. B. Stouffer, Eula A. Weeks, W. D. A. Westfall, Jessica M. Young.

On Friday evening at the American Hotel Annex a dinner was held jointly with the Southwestern Section of the American Mathematical Society and the Central Association of Science and Mathematics Teachers. Father W. J. Ryan, vice-president of St. Louis University and a member of the Association, acted as toastmaster at the dinner, and the following addresses were given: "The age of power" by Mr. A. S. Langsdorf, formerly dean of the Schools of Engineering and Architecture of Washington University; "Zoölogy in the secondary schools" by Dr. Caswell Grave, professor of zoölogy, Washington University.

At noon on Saturday the members of the Missouri Section of the Association and the Southwestern Section of the Society were the guests of Washington University at a luncheon which was served in the Tower Dormitory dining hall.

The following officers were elected for the ensuing year: Chairman, E. R. HEDRICK, University of Missouri; Vice-Chairman, W. A. LUBY, Kansas City Junior College; Secretary-Treasurer, P. R. RIDER, Washington University.

The 1922 meeting will be held in Kansas City in November, at the time of the meeting of the Missouri State Teachers' Association.

The following papers were read:

(1) "Mathematics clubs in junior high schools" by Mr. A. H. HUNTINGTON, Cleveland High School, St. Louis;

(2) "Some suggestions in regard to mathematics" by Father W. J. RYAN, vice-president of St. Louis University;

(3) "Correct methods of making drawings of space objects" by Professor W. H. ROEVER, Washington University;

(4) "The relation of mathematics to engineering" by Professor E. R. HEDRICK, University of Missouri;

(5) "Graphical methods of representing a function of a function and of solving allied problems" by Professors HEDRICK and ROEVER;

(6) "An elementary exposition of the theorem of Bernoulli with applications to statistics" by Professor H. L. RIETZ, University of Iowa;

(7) "Final report of the National Committee on Mathematical Requirements" by Dr. EULA A. WEEKS, Cleveland High School, St. Louis.

In addition to these papers, an informal talk was given by Professor H. E. SLAUGHT, of the University of Chicago, who told the Section of the recent grant to the Association by Mrs. Paul Carus of a sum of money to be used for the publication of expository monographs on mathematical subjects. In the absence of the author, the paper by Professor Rietz was read by Professor C. H. ASHTON of the University of Kansas. Several of the papers led to interesting discussions. Abstracts of the papers follow below, the numbers corresponding to the numbers in the list of titles:

1. Mr. Huntington discussed mathematics clubs for pupils of junior high school age, maintaining that they offer opportunities not yet realized for enlisting the interest and effort of boys and girls in the study of mathematics.

2. Father Ryan offered some suggestions in regard to the present day status of mathematics in the high schools, which led up to a discussion of measures that should be taken to promote intensive work in mathematics.

3. In this paper Professor Roever first stated the criteria which must be satisfied by a drawing (on a plane or curved surface) in order that it should be an adequate and good looking representation of a space object. He then showed, as a consequence of these criteria, that a drawing must be a central or parallel (orthographic or oblique) projection of the space object to be represented, and gave the conditions under which one or two projections are required in order adequately to represent the object. Finally, rules of procedure were developed and exemplified.

4. Professor Hedrick called the attention of the Section to the organization of a joint committee under the auspices of the Society for the Promotion of Engineering Education and the Mathematical Association of America. He pointed out the close relationship between mathematics and engineering, and mentioned briefly both the history of previous investigations of mathematical teaching in schools of engineering and the plans of the present committee.

5. Professors Hedrick and Roever called attention to a recent paper (R. von Huhn, "A new graphic analytic method," *Science*, October 7, 1921) and pointed out that it is a very special case of the general method given by Professor Roever in this MONTHLY, *1916*, 330–333. They also showed how the method could be used in more general problems, including the elimination of one variable from two equations in three variables. [Published in *Science*, April 14, 1922.]

6. In this paper Professor Rietz showed by concrete illustrations the character of the statistical problems to which the theorem of Bernoulli is applicable, and presented an elementary demonstration of the theorem for the purpose of giving a clear view of a method of treatment of a fundamental problem in the fluctuations of results derived from random samples. [Published in *Mathematics Teacher*, December, 1921.]

7. The paper of Dr. Weeks, who is a member of the National Committee on Mathematical Requirements, gave a brief outline of the contents of the final report of the committee, with statements about the publication and distribution of this report.

PAUL R. RIDER, *Secretary-Treasurer.*

THE NOVEMBER MEETING OF THE TEXAS SECTION.

The first regular meeting of the Texas Section was held at the Bryan High School, Dallas, Texas, on November 25–26, in conjunction with a meeting of the Mathematics Section of the Texas State Teachers' Association. Three meetings were held. G. C. EVANS presided at the session of Friday morning, H. J. ETTLINGER, president of the Mathematics Section of the Texas State Teachers' Association, at the session of Friday afternoon, and J. O. MAHONEY at the closing session of Saturday morning.

One of the very enjoyable features of the occasion was the banquet for the members of the Texas Section of the Association held in the Violet Room of the Oriental Hotel. President Ettlinger presided as toastmaster.

There were sixty-five in attendance, including the following eighteen members of the Association:

W. D. Baten, H. Y. Benedict, A. A. Bennett, J. E. Burnam, P. J. Daniell, H. J. Ettlinger, G. C. Evans, L. R. Ford, A. J. Hargett, H. Hosford, E. H. Jones, J. O. Mahoney, R. L. Moore, H. Porter, J. L. Riley, P. C. Rockwell, E. R. Tucker, P. H. Underwood, N. Wunder.

The following officers were elected for the coming year: Chairman, G. C. EVANS; Vice-Chairman, A. A. BENNETT; Secretary-Treasurer, W. D. BATEN. The next meeting will be held at Houston, December 1–2, 1922, in conjunction with the annual meeting of the Mathematics Section of the Texas State Teachers' Association.

The following papers were read:

(1) "Dante and science" by Professor G. C. EVANS;

(2) "What the college or university expects of the freshman in the way of mathematics" by Professor H. J. ETTLINGER;

(3) "An actuary: his training, functions and service" by C. P. ROCKWELL;

(4) "The rôle of logic in the refinement of geometric intuition" by Professor R. L. MOORE;

(5) "What the state section of the Mathematical Association of America can do for mathematics in Texas" by Professor J. L. RILEY;

(6) "On teaching the fractional exponent" by Professor W. D. BATEN;

(7) "The number system of algebra" by Professor A. A. BENNETT.

Abstracts of most of the papers and discussions follow below, the numbers corresponding to numbers in the list of titles.

1. Professor Evans gave a résumé of his paper as published in *The Rice Institute Pamphlet*, volume 8, April, 1921, on the occasion of the Dante Sexcentenary. Before his exile Dante had demonstrated his extraordinary ability and his interest in both poetry and science. No reader can follow adequately a work like the *Divine Comedy* without some knowledge of the mechanical frame on which it is consciously built, and some understanding of the physical aspect of the mediæval universe. For here Dante himself was an expert. The astronomy of Dante's time, which occupied a secondary place to that of astrology, was merely the astronomy of Hipparchus. Interpreted in the light of modern knowledge it was an approximate representation of apparent angular motions of the stars and planets as seen from the earth, by means of a system of uniformly moving epicycles and eccentrics. It was thus an attempt to build up the variable apparent motions of the heavens out of a system of uniform circular motions about centers themselves also moving uniformly.

Knowing well the phenomena of the precession of the equinoxes discovered by Hipparchus, Dante was enabled to proceed accurately on his voyage of the heavens. According to Dante the earth occupies a central position at rest and

the orbits of the moon, planets, sun, and stars are fixed on concentric spheres, or heavens of various radii.

The mediæval, or Dantean, order of the spheres is this: (1) Moon, (2) Mercury, (3) Venus, (4) Sun, (5) Mars, (6) Jupiter, (7) Saturn, (8) Stars, (9) Crystalline or Primum Mobile, (10) the Empyrean Heaven.

2. Professor Ettlinger gave the conclusions drawn from the answers to a questionnaire concerning the knowledge of mathematics required in other departments of the University of Texas of students who are taking or have completed the work of the freshman year. Opinions were solicited from each department relative to the value of each topic listed. The plan of the questionnaire followed the fourth section of that issued by the National Committee on Mathematical Requirements. This paper will be published in the *Texas Mathematics Teachers' Bulletin*.

3. Mr. Rockwell, in the course of his remarks, said: "Mathematical training, with patient study and practice, may make a perfect mathematician, but still fail to give us an actuary. To the born mathematical genius, or to the person with acquired mathematical talent, must be added something that is absolutely essential to produce the actuary. That something includes a minute observation of the practical workings of life insurance, the ability to recognize and appreciate the weight and importance of business interests and business requirements."

Recent features of insurance practically new to American life, as group insurance, industrial insurance and annuities, have increased the responsibilities of the actuary. The generally accepted notion that the sole duty of the actuary is to furnish tables in keeping with the "policies" by the company agents has been dispelled. The actuary must be in close coöperation with the medical director so as to pass upon certain risks and with the general directors to advise on certain financial risks and policies.

4. Professor Moore spoke of the rôle of logic in the treatment of propositions in elementary geometry. The demonstration of some well-known elementary propositions led to rather fanciful results, or fallacies, when demonstrated without regard to logical sequence. The age-old problem of whether or not strictly logical treatment must be required was discussed.

6. Professor Baten spoke of the troubles that beset first year students when they take up the work of exponents in college algebra. He emphasized the very close relation existing between the study of exponents, the study of logarithms and the subject of radicals. The fact that the average freshman did not know that the value of $a^0 = 1$ was reasserted.

7. In this paper, Professor Bennett discussed the development of the number system of algebra through its principal historical and logical steps. He contrasted the usual method of elementary arithmetic, wherein the discussion of fractions precedes that of negative numbers, with the procedure in algebraic treatments, where this sequence is usually reversed, and spoke of the justification of each line of development. The principal feature of the paper was an analysis of the postulates of algebra into three groups, the first treating of rational operations

alone, the second, of order relations alone, the third, of order relations among the results of rational operations. This classification corresponds intimately to the characterizations of the theorems of algebra as falling into three analogous categories.

J. L. RILEY, *Secretary-Treasurer.*

CUSPIDAL ENVELOPE ROSETTES.

By WILLIAM F. RIGGE, Creighton University.

A point P moves in the line segment EG, Fig. 1, with simple harmonic motion of p cycles, while this segment makes q revolutions about A with uniform angular speed. Moritz[1] has exhaustively treated the case when the point A is in the line EG or in its prolongation. The writer[2] has shown that when the point A is out of the line EG and the rosette drawn is cuspidal, AL, the distance of EG from A, must be $n \sin \beta$ (in which $n = p/q$) and LR, the distance of R, the mid-point, or point of zero phase, of EG, from its point of tangency L on the *tangent* circle, must be $\cos \beta$. The point P remains on an ellipse whose conjugate semi-axis is unity ($= ER = RG$) and is always parallel to EG, whose major semi-axis $= n$, and whose center is the sine PR of the phase α distant from A, β being the eccentric angle of P.

When the point P is given a double rectilinear harmonic motion[3] with equal amplitudes but with periods in the ratio of m to 1, it may be conceived to move with simple harmonic motion of mn periods on the line segment DF, Fig. 2, while this line slides n times in a similar way along the line EG in one revolution of EG about A. For the sake of greater clearness these lines DF and EG are spaced a short distance apart in the figure. PS is then the sine of the phase of P on DF, while RS is the sine of the phase of S, the mid-point of DF, on EG. Hence the distance of the tracing point P from the *tangent* circle measured along the tangent line $GFED$ is

$$LP = -RL + RS + SP = -\cos\beta + \sin n\theta + \sin mn\theta,$$

in which θ is the phase of the circular motion about A. In the previous paper the point A was in the line EG. In the present A will be out of this line. The discussion will, as before, be confined to envelopes that are cuspidal.

Two Envelopes. As the points D and F are the limits of the excursions of P on this line segment DF, it is clear that these points themselves would trace the envelopes to all the lobes or loops or branches drawn by P, and that P must be on these envelopes when it is in phases 90° and 270°, respectively, on DF. The

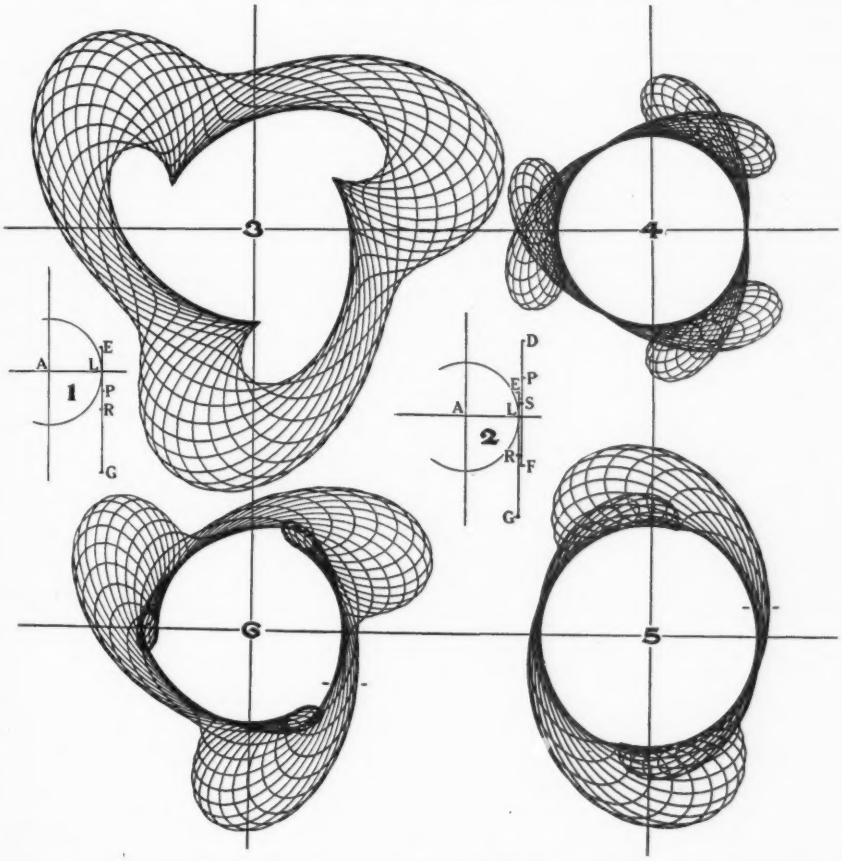
[1] "On the construction of certain curves in polar coördinates" by R. E. Moritz, in this MONTHLY, *1917*, 213–220.

[2] "Concerning a new method of tracing cardioids" by W. F. Rigge, in this MONTHLY, *1919*, 21–32. "Cuspidal rosettes," in this MONTHLY, *1919*, 332–340.

[3] "Envelope Rosettes," in this MONTHLY, *1920*, 151–157.

distance between the two envelopes measured in the line of motion of P is thus equal to 2, and the distance of P from them when in phase 0° or 180°, or multiples of them, must be -1 and $+1$.

The Starting Position of P for Cuspidal Envelopes. If D (or F) is to trace a cuspidal rosette, P must be set on a point of this rosette in phase 90° (or 270°) on DF, and then the phase of S on EG must also be set according to the nature of the curve. As the phase of S on EG, when $m - 1$ is infinitesimal, may have



any value corresponding to any phase of P on DF, we select the convenient value of $\theta = 0°$, when both will be zero together, to start P moving. Then P, R, S are coincident in Fig. 2. But as the distance of P from the D (or F) envelope will then be $+1$ or -1, this starting position of P, R, S, in phase 0° must therefore be the distance unity on one side or other of the cuspidal rosette, that is, its starting ordinate $y_0 = -\cos\beta - 1$ or $-\cos\beta + 1$, while x_0 is always

$n \sin \beta$. From this it is evident that only one of the D and F envelopes can be cuspidal, except in a special case to be mentioned presently, when both may be so.

The Influence of β. When y_0 or $-\cos\beta \pm 1$ is small numerically, the D and F envelopes will as a rule intersect. Although unequal, they are always symmetrical on account of the positions of E and G on opposite sides of L. They are equal only when $y_0 = 0$, that is, when $\beta = 0°$ or $180°$, being coincident when n is odd, and symmetrically displaced when n is even. When y_0 or $-\cos\beta \pm 1$ is large numerically, the D and F envelopes cannot intersect, because E and G are then at very unequal distances from L and A. The inner one of the two will be cuspidal.

Illustrations. Fig. 3 with $n = 3$, $\beta = 30°$, $x_0 = n \sin\beta = 1.5$, $y_0 = -\cos\beta - 1 = -1.866$, shows the D or inner envelope cuspidal, while Fig. 4 with the same values of n, β, x_0, but with $y_0 = -\cos\beta + 1 = +0.134$, would seem at first sight to show two equal symmetrical and cuspidal envelopes. Only one however is in fact cuspidal and congruent to the inner envelope of Fig. 3. It is here the F envelope and is really smaller than the other or D envelope, because in this case $GL < EL$. The D envelope of Fig. 3 is in every way exactly equal to the F envelope of Fig. 4. When $\beta = 60°$ (but $n = 2$), as in Fig. 5, the inequality of the two envelopes is obvious, the smaller alone being cuspidal.

Transition Envelopes. Fig. 6 shows a transition envelope for $n = 3$. When the tracing point P is started in phases $0°$ with a smaller numerical value of y_0 than $-\cos\beta - 1$ as in Fig. 3, the cusp throws a lobe or a shoot like a growing bud, while the outer envelope is contracted. While y_0 is increasing in the positive direction, the two envelopes may become apparently equal or nearly so, although neither can be cuspidal. The cuspidal stage will be reached again when $y_0 = -\cos\beta + 1$ as in Fig. 4. After that the growth just mentioned will be reversed, until the value of $\beta' = 180° - \beta$ will again give cuspidal envelopes symmetrical however to those of β. For still larger positive values of y_0, the envelopes will tend to become nearly equal and circular.

Cuspidal envelope rosettes are best drawn when n is a small integer. When $n = 1$ one or both of the envelopes are cardioids for all values of β.

SEXAGESIMAL FRACTIONS AMONG THE BABYLONIANS.

BY FLORIAN CAJORI, University of California.

In view of the fact that certain writers have expressed the conviction[1] that, while the Babylonians operated with integers expressed in the sexagesimal system, they *did not use sexagesimal fractions*, it is worth while to refer briefly to a cuneiform tablet recently described by H. F. Lutz[2] of the University of Pennsylvania which unquestionably reveals the Babylonian use of sexagesimal frac-

[1] For example, see this MONTHLY, *1920*, 124.

[2] *American Journal of Semitic Languages and Literatures*, vol. 36, 1920, pp. 249–257. Tablet CBS 8536 in the University Museum, Philadelphia.

tions. According to Lutz, the tablet "cannot be placed later than the Cassite period, but it seems more probable that it goes back even to the First Dynasty period, *ca.* 2000 B.C."

To mathematicians the tablet is of interest because it reveals operations with sexagesimal fractions resembling modern operations with decimal fractions. For example, 60 is divided by 81 and the quotient expressed sexagesimally. Again, a sexagesimal number with two fractional places, 44(26)(40), is multiplied by itself, yielding a product in four fractional places, namely [32]55(18)(31)(6)(40). In this notation the [32] stands for 32×60 units, and to the (18), (31), (6), (40) must be assigned, respectively, the denominators 60, 60^2, 60^3, 60^4.

Numbers that are incorrect are marked by a *.

First column		*Fifth column*	
. . . gal (?) -bi 40 -ám			
šu *a- na* gal-bi 30 -ám			
			44(26)(40)
igi 2	30	1	44(26)(40)
igi 3	20	2	[1]28(53)(20)
igi 4	15	3	[2]13(20)
igi 5	12	4	[2]48(56)(40)*
igi 6	10	5	[3]42(13)(20)
igi 8	7(30)	6	[4]26(40)
igi 9	6(40)	7	[5]11(6)(40)
igi 10	6	9	[6]40
igi 12	5	10	[7]24(26)(40)
igi 15	4	11	[8]8(53)(20)
igi 16	3(45)	12	[8]53(20)
igi 18	3(20)	13	[9]27(46)(40)*
igi 20	3	14	[10]22(13)(20)
igi 24	2(30)	15	[11]6(40)
igi 25	2(24)	16	[11]51(6)(40)
igi 28*	2(13)(20)	17	[12]35(33)(20)
igi 30	2	18	[13]20
igi 35*	1(52)(30)	19	[14]4(26)(40)
igi 36	1(40)	20	[14]48(53)(20)
igi 40	1(30)	30	[22]13(20)
igi 45	1(20)	40	[29]37(46)(40)
igi 48	1(15)	50	[38]2(13)(20)*
igi 50	1(12)	44(26)(40)a-na 44(26)(40)	
igi 54	1(6)(40)	[32]55(18)(31)(6)(40)	
igi 60	1	44(26)(40) square	
igi 64	(56)(15)	igi 44(26)(40)	81
igi 72	(50)	igi 81	44(26)(40)
igi 80	(45)		
igi 81	(44)(26)(40)		

The tablet contains twelve columns of figures. The first column gives the results of dividing 60 in succession by twenty-nine different divisors from 2 to 81. The eleven other columns contain tables of multiplication; each of the numbers 50, 48, 45, 44(26)(40), 40, 36, 30, 25, 24, 22(30), 20 is multiplied by integers up to 20, then by the numbers 30, 40, 50, and finally by itself. Using our modern numerals, we reproduce the first and the fifth columns of the tablet, which exhibit a larger number of fractions than do the other columns. The Babylonians had

no mark separating the fractional from the integral parts of a number. Hence a number like 44(26)(40) might be interpreted in different ways; among the possible meanings are $44 \times 60^2 + 26 \times 60 + 40$, $44 \times 60 + 26 + 40 \times 60^{-1}$, and $44 + 26 \times 60^{-1} + 40 \times 60^{-2}$. Which interpretation is the correct one can be judged only by the context, if at all.

The exact meaning of the first two lines in the first column is uncertain. In this column 60 is divided by each of the integers written on the left. The respective quotients are placed on the right.

In the fifth column the multiplicand is 44(26)(40) or 44 4/9. The last two lines seem to mean "$60^2 \div 44(26)(40) = 81$, $60^2 \div 81 = 44(26)(40)$."

It is a source of gratification to find that scholars of several thousand years ago were fully as capable of committing errors in computation, as are arithmeticians of the present time.

The Babylonian use of sexagesimal fractions is shown also in a clay tablet described by A. Ungnad[1] (*Orient. Lit. Zeitung*, 19 Jahrg., 1916, p. 363-368). In it the diagonal of a rectangle whose sides are 40 and 10 is computed by the approximation $40 + 2 \times 40 \times 10^2 \div 60^2$, yielding 42(13)(20), and also by the approximation $40 + 10^2 \div \{2 \times 40\}$, yielding 41(15). Translated into the decimal scale, the first answer is 42. 22 +, the second is 41.25, the true value being 41.23 +.

A BUDGET OF EXERCISES ON DETERMINANTS.

By THOMAS MUIR, Rondebosch, South Africa.

A collection of fresh exercises[2] on a mathematical subject, even if the plan and execution be far from perfect, can be of greater service to the student than any so-called "paper," covering the same amount of page space. It is in this belief that I have brought together the following budget of thirty. In a kind of way they range over the whole subject of determinants: at any rate they do not confine themselves to any special branch of it. They are of all degrees of difficulty, starting with commonplace instances of mere "evaluation." They naturally also differ in suggestiveness; one or two of them might in eager hands lead to the formulating of allied results, and thereby even to the evolution of material for a "paper." None of them, so far as I can at present recall, has been printed before, and certainly, the number of them that may so turn out must be comparatively trifling.

1. If $\begin{vmatrix} a_1 - x & a_2 - x & a_3 - x \\ b_1 - x & b_2 - x & b_3 - x \\ c_1 - x & c_2 - x & c_3 - x \end{vmatrix} = 0$, then $x = -\,|a_1 b_2 c_3| \div \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & a_1 & a_2 & a_3 \\ 1 & b_1 & b_2 & b_3 \\ 1 & c_1 & c_2 & c_3 \end{vmatrix}$.

[1] Our information is drawn from *Mitteilungen zur Geschichte der Medizin und der Naturwissenschaften*, Leipzig, vol. 17, 1918, p. 203.

[2] For any technical terms, in the following, which the student may have doubt about he is referred to the text-books of Weld and Hanus.

2. $$\begin{vmatrix} 1 & \sin(\beta+\gamma) & \cos(\beta-\gamma) \\ 1 & \sin(\gamma+\alpha) & \cos(\gamma-\alpha) \\ 1 & \sin(\alpha+\beta) & \cos(\alpha-\beta) \end{vmatrix} = \begin{vmatrix} \cos 2\alpha & \cos\alpha & \sin\alpha \\ \cos 2\beta & \cos\beta & \sin\beta \\ \cos 2\gamma & \cos\gamma & \sin\gamma \end{vmatrix}.$$

3. $$\begin{vmatrix} \cos(\beta+\gamma) & \sin(\beta+\gamma) & \cos(\beta-\gamma) \\ \cos(\gamma+\alpha) & \sin(\gamma+\alpha) & \cos(\gamma-\alpha) \\ \cos(\alpha+\beta) & \sin(\alpha+\beta) & \cos(\alpha-\beta) \end{vmatrix} = \tfrac{1}{2}\begin{vmatrix} \sin 2\alpha & \cos 2\alpha & 1 \\ \sin 2\beta & \cos 2\beta & 1 \\ \sin 2\gamma & \cos 2\gamma & 1 \end{vmatrix}.$$

4. Any axisymmetric determinant is expressible as the product of its principal term and an axisymmetric determinant with a diagonal of units.

5. In an n-line zero-axial determinant the number of positive terms is $n-1$ greater or less than the number of negative terms according as n is odd or even.

6. The reversible continuant $K(x^a x^b x^c x^c x^b x^a x)$ of odd order is resolvable into two continuants.

7. Express the square of the persymmetric determinant

$$\begin{vmatrix} a & b & c & d \\ b & c & d & am \\ c & d & am & bm \\ d & am & bm & cm \end{vmatrix}$$

as a persymmetric.

8. $$|(a_r + s - 1)^m|_{m+1} = 1^1 2^2 \cdots m^m \zeta^{1/2}(a_1, a_2, \cdots, a_{m+1}).$$

What is written here between the two vertical lines is the (r, s)th element of the determinant, and the suffix outside denotes the order: the first row is thus

$$a_1^m, \qquad (a_1+1)^m, \qquad \cdots, \qquad (a_1+m)^m.$$

When instead of the suffix $m+1$ we put n, we are faced with a more complicated problem. $\zeta^{1/2}(a_1, a_2, \cdots, a_{m+1})$ denotes the difference product of the a's.

9. If $A = (b+c)(b+d)(c+d)$, $B = (a+c)(a+d)(c+d)$, $\cdots$, then

$$\zeta^{1/2}(A, B, C, D) = (a+b+c+d)^6 \cdot \zeta^{1/2}(a^2, b^2, c^2, d^2).$$

10. If Δ stand for $|a_1 b_2 c_3 d_4|$, then

$$\begin{vmatrix} A_1 - \dfrac{\Delta}{a_1} & A_2 & A_3 & A_4 \\ B_1 & B_2 - \dfrac{\Delta}{b_2} & B_3 & B_4 \\ C_1 & C_2 & C_3 - \dfrac{\Delta}{c_3} & C_4 \\ D_1 & D_2 & D_3 & D_4 - \dfrac{\Delta}{d_4} \end{vmatrix} = \frac{\Delta^3}{a_1 b_2 c_3 d_4} \begin{vmatrix} 0 & a_2 & a_3 & a_4 \\ b_1 & 0 & b_3 & b_4 \\ c_1 & c_2 & 0 & c_4 \\ d_1 & d_2 & d_3 & 0 \end{vmatrix}.$$

11. $$\begin{vmatrix} abcd & abd & acd & bcd \\ abc & abcd & acd & bcd \\ abc & abd & abcd & bcd \\ abc & abd & acd & abcd \end{vmatrix} = a^3 b^3 c^3 d^3 (abcd - \Sigma ab + 2\Sigma a - 3)$$

and the cofactor of $a^3b^3c^3d^3$ on the right is expressible as a continuant and also as a recurrent. What does this cofactor become when the number of variables and the order of the determinant are each made n?

12. In a 3-by-5 array whose row-sums vanish there are three independent linear relations between the determinants of the array.

13.
$$\begin{vmatrix} x_1 & a_2 & a_3 & a_4 & a_5 \\ -a_1 & x_2 & a_3 & a_4 & a_5 \\ -a_1 & -a_2 & x_3 & a_4 & a_5 \\ -a_1 & -a_2 & -a_3 & x_4 & a_5 \\ -a_1 & -a_2 & -a_3 & -a_4 & x_5 \end{vmatrix} = \begin{vmatrix} x_1 & -1 & 0 & 0 & 0 \\ a_1a_2 & x_2 & -1 & 0 & 0 \\ a_1x_2a_3 & a_2a_3 & x_3 & -1 & 0 \\ a_1x_2x_3a_4 & a_2x_3a_4 & a_3a_4 & x_4 & -1 \\ a_1x_2x_3x_4a_5 & a_2x_3x_4a_5 & a_3x_4a_5 & a_4a_5 & x_5 \end{vmatrix}$$
$$= x_1x_2x_3x_4x_5 + \Sigma x_1x_2x_3a_4a_5 + \Sigma x_1a_2a_3a_4a_5.$$

14. Simplify the determinant whose first three columns are taken from those of any 4-by-5 array and whose other column is a column taken from the square of the array.

15. Show that the cofactor of c in

$$\begin{vmatrix} 0 & a & b & c \\ l & 1 & x & yz - x^2 \\ m & 1 & y & zx - y^2 \\ n & 1 & z & xy - z^2 \end{vmatrix}$$

is a factor of the determinant, and find the other factor.

16. Any positive unit orthogonant $|\alpha_1\beta_2\gamma_3|$ is axisymmetric if $1+\alpha_1+\beta_2+\gamma_3$ vanishes or if $\alpha_1 = \beta_2 = \gamma_3 = 1$.

17. If Δ stand for $ad - bc$, then

$$\frac{\partial(a\Delta, b\Delta, c\Delta, d\Delta)}{\partial(a, b, c, d)} = 3\Delta^4.$$

18. The bordered axisymmetric determinant

$$\begin{vmatrix} 0 & x & y & z \\ u & -d+e+f & f & e \\ v & f & d-e+f & d \\ w & e & d & d+e-f \end{vmatrix} = \begin{vmatrix} x & y & z \\ & d-e & d-f \\ & & e-f \end{vmatrix} \cdot \begin{vmatrix} u & v & w \\ & d-e & d-f \\ & & e-f \end{vmatrix}$$

and therefore equals the bordered skew

$$\begin{vmatrix} 0 & x & y & z \\ -u & 0 & d-e & d-f \\ -v & e-d & 0 & e-f \\ -w & f-d & f-e & 0 \end{vmatrix}.$$

19. Eliminate x, y, z, w from the equations got by putting $r = 1, 2, 3, 4$ in

$$a_rxy + b_rxz + c_rxw + d_ryz + e_ryw + f_rzw = 0.$$

20. If

$$\begin{aligned} ax + by + cz &= 0, \\ by + du + ev &= 0, \\ cz + ev + fw &= 0, \end{aligned} \qquad \text{and} \qquad \begin{aligned} mx + ny + pz &= 0, \\ ny + qu + rv &= 0, \\ pz + rv + sw &= 0, \end{aligned}$$

then

$$(mc - ap)(bq - dn)(es - rf) = (dr - qe)(pf - cs)(an - bm).$$

21. $$\begin{vmatrix} |a_2b_3| & |a_1b_3| & |a_1b_2| \\ |c_2x_3| & |c_1x_3| & |c_1x_2| \\ |y_2c_3| & |y_1c_3| & |y_1c_2| \end{vmatrix} = \begin{vmatrix} |b_2c_3| & |b_1c_3| & |b_1c_2| \\ |a_2c_3| & |a_1c_3| & |a_1c_2| \\ |x_2y_3| & |x_1y_3| & |x_1y_2| \end{vmatrix} = |a_1b_2c_3|\,|c_1x_2y_3|.$$

22. If P be the product of n linear homogeneous functions of n variables, then (H standing for Hessian)

$$\frac{H(P)}{H(\log P)} = -(n-1)P^n.$$

23. If the determinant whose columns are the lth, mth, nth, rth columns of any 4-by-8 array be denoted by $|lmnr|$, then

$$\begin{vmatrix} |1567| & |1568| & |1578| & |1678| \\ |2567| & |2568| & |2578| & |2678| \\ |3567| & |3568| & |3578| & |3678| \\ |4567| & |4568| & |4578| & |4678| \end{vmatrix} = |1234| \cdot |5678|^3.$$

24. $$\begin{vmatrix} ax & bx+\beta & cx+\gamma \\ bx-\beta & ex & dx+\delta \\ cx-\gamma & dx-\delta & fx \end{vmatrix} = \begin{vmatrix} ax & a\beta & M \\ -a\beta & aPx & N \\ -M & -N & PQx \end{vmatrix} \div a^2P^2,$$

where

$$P = \begin{vmatrix} a & b \\ b & e \end{vmatrix}, \qquad Q = \begin{vmatrix} a & b & c \\ b & e & d \\ c & d & f \end{vmatrix}, \qquad M = \begin{vmatrix} a & b & c \\ b & e & d \\ 0 & \beta & \gamma \end{vmatrix},$$

$$N = \begin{vmatrix} 0 & a & b & c \\ 0 & b & e & d \\ a & 0 & \beta & \gamma \\ b & -\beta & 0 & \delta \end{vmatrix}.$$

25. $$\begin{vmatrix} 0 & a_1b_2+a_2b_1 & a_1b_3+a_3b_1 & a_1b_4+a_4b_1 & \cdots \\ a_1b_2+a_2b_1 & 0 & a_2b_3+a_3b_2 & a_2b_4+a_4b_2 & \cdots \\ a_1b_3+a_3b_1 & a_2b_3+a_3b_2 & 0 & a_3b_4+a_4b_3 & \cdots \\ a_1b_4+a_4b_1 & a_2b_4+a_4b_2 & a_3b_4+a_4b_3 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_n$$

$$= \left\{ -n + 1 - \sum \frac{|a_1b_2|^2}{(-2a_1b_1)(-2a_2b_2)} \right\} \prod (-2a_1b_1).$$

26. The sum of a quadric and its discriminant is expressible as a Pfaffian.

27. The determinant of the sum of an axisymmetric matrix and a zero-axial skew matrix is expressible as a Pfaffian.

28. If D_1, D_2 be n-line determinants, the one axisymmetric and the other zero-axial skew, and Δ_1, Δ_2, $\cdots$ be the series of determinants each of which is identical in m columns with D_2 and in the remaining columns with D_1, then when m is odd

$$\sum\Delta = 0.$$

29. If from the n-line determinants D_1, D_2, $\cdots$, D_p, there be formed a new n-line determinant Δ consisting of α_1 rows from D_1, α_2 rows from D_2, and so on, the rows in each case occupying the same places in Δ as in their own determinant, then

$$\sum\Delta = \sum\Delta',$$

where Δ' is the determinant formed like Δ save that columns are used instead of rows.

30. If A and O be n-line determinants, A axisymmetric and O orthogonant, the sum of the r-line coaxial minors of OAO' is equal to the corresponding sum in A.

Δ' is used to denote the conjugate of Δ, *i.e.*, the determinant got by changing the rows of Δ in order into columns.

February 15, 1921.

AMONG MY AUTOGRAPHS.

By DAVID EUGENE SMITH, Columbia University.

18. Sylvester as a Poet.

The natural relation of the mathematical to the poetical mind has been observed so often as to make it hardly worth while to comment upon it. It is probably not too much to say that every mathematician is a poet at heart, and it is possible that every poet has more of the mathematical in his soul than he thinks. However this may be, it would not be without interest to study the tangible evidence of the relation of poetry to mathematics, and, indeed, of mathematics to all the various arts in which rhythm and symmetry and imagination enter. In such a study the productions of that interesting character, Professor Sylvester, would naturally be considered—not because they represent a high type of imaginative literature, but because they illustrate an interesting type of mind.

Among a large number of autograph letters of Professor Sylvester which I possess is one written on November 9, 1884, to W. J. C. Miller, Esq., mathematical editor of the *Educational Times*. This contains a poem entitled "Retrospect," which was published a little later. It shows the alterations which Professor Sylvester made and which, having never before appeared in print, may be welcomed by the few who have already seen the verses. The poem as a whole may be of interest to those who have attempted to solve what may be designated as Professor Sylvester's "other" problems. The verses and accompanying note are as follows:

RETROSPECT.

'Ere told it's message, spent
The warning flash at sea,
Stars through dim firmament
Making mad revelry;

Before earth's[1] sovereign eye
Night's silver mantle drawn,
With sobering alchemy
In pale day quenched the dawn;

Strange pause in merriment,
The hush of minstrelsy,
Silence more eloquent
Than tuneful threnody,

Life's bootless endless chain
Things born to cease to be
Pleasure subserving[2] pain
Thought bitter[3] reverie,

Loved lost Eurydice
Snatched to redeemless death,
Her spouse distraught to see
And suck once more her breath,—

What tales of eld rehearse,
Whisper these signs to me,
"Hope lights to swift reverse
Death is faith's destiny."

Dr Mr Miller

These are the verses referred to in my previous note in their more perfect form. The *third* stanza is new[4] the 18th and 19th lines altered and the 22nd retouched.

Yours truly,

J. J. S.

9th Novr.[5] 1884.

19. LEWIS CARROLL AS A CRITIC.

To those who have read *Alice in Wonderland* (and who has not?) or Mr. Stuart Collingwood's *Life of Lewis Carroll* (which everyone should do), personal letters of Mr. Dodgson can hardly fail to be of interest. Mr. Collingwood has given many letters in his biography, but in general they are such as were written to the child friends of Dodgson,—or, to use his literary name, of Lewis Carroll. Among those of his letters which are now in my collection is one which may have some interest because it shows Dodgson as the crusty bachelor that he was, and hence in a somewhat different light. It was written when he was fifty-nine years old, while he was still engaged in his tutoring at Christ Church, Oxford (where he had matriculated forty-one years before), and nearly thirty years after he began his various adventures of Alice, culminating in the children's

[1] In the manuscript "night's" is replaced by "earth's."
[2] "Subserving" replaces "the mask to."
[3] "Bitter" replaces "thrall to."
[4] "and and" erased after "new."
[5] "Novr." replaced "Oct.," showing his mental lapse.

well-known classic which appeared in 1865. It is not improbable that his remarks relative to editorial revision will strike a responsive note in the soul of other writers, and possibly of certain editors as well. At any rate, my friend, the late W. J. C. Miller, to whom it was written, laughed over it when he gave it to me, many years ago.

The letter is as follows:

Ch. Ch. Oxford.
Feb. 21/91

Dear Sir,

I have carefully examined "The Foundations of Geometry" by E. T. Dixon, published by Deighton, Bell, & Co., Cambridge, and would be glad to send you a short notice of it, if you would be at all likely to print it. You would be free to reject it, of course; & also to abridge it, if you thought fit: the only condition I make is that no word shall be *altered*, or *added*. Also you would be free to put my name to it, or not, as you thought best—The latter would be the best, in my own opinion. I think the notice would have more weight, & get more attention, as an unsigned 'notice' than as a signed letter.

Kindly tell me whether to send it or not. I enclose a card for your reply.

Truly yours,
C. L. DODGSON.

The Mathematical Editor of
"The Educational Times."

P.S. The book is on the "direction" theory, and involves, I believe, the same logical fallacy as lies at the root of Wilson's treatise.

RECENT PUBLICATIONS.

REVIEWS.

Geschichte der Elementar-Mathematik in systematischer Darstellung mit besonderer Berücksichtigung der Fachwörter. Von J. TROPFKE. Erster Band, Rechnen. Zweite, verbesserte und sehr vermehrte Auflage. Berlin and Leipzig, Vereinigung Wissenschaftlicher Verleger, 1921. 8vo. 7 + 177 pages. Price, to Germans, bound, 46 marks.

The first edition of Tropfke's history, in two volumes, 1902–1903, made a notable contribution to the subject by its clarity of statement, and wealth of exact references to sources. The two parts of the first volume were devoted to "Das Rechnen" (pp. 1–122) and "Die Algebra" (pp. 123–332). The twelve parts of the second volume (496 pages) dealt with geometry, logarithms, plane and spherical trigonometry, spherical geometry, stereometry, series, compound interest, partial fractions, analytic geometry, conic sections, maxima and minima, combinatory analysis and theory of probability.

But the numerous discoveries of the past twenty years have rendered many of the statements of this work inaccurate and otherwise inadequate. A thorough revision was therefore very much to be desired. The previous two-volume edition, costing 22 marks, is to be expanded somewhat and published in seven volumes, costing 322 marks—if the cost of the volumes is to be uniform: I, Das Rechnen; II, General arithmetic; III, Proportion and equations; IV, Geometry; V, Trigonometry and spherics; VI, Analysis; VII, Name and subject indexes.

The first volume of the new edition covers the first 120 pages of the old edition; there is thus an increase in the number of pages of more than 45 per cent. The expansions are mainly in connection with footnotes, more than doubled in number, and with the sections on "Numbers in general" and "Integers." The treatment of perfect numbers in this latter-named section would have been vastly improved if due account had been taken of the comprehensive survey in Dickson's *History of the Theory of Numbers*, volume 1, 1919, (see *1921*, 140–141). Tropfke gives no reference to results obtained regarding odd perfect numbers, and the twelve known perfect numbers are, therefore, without indicated warrant called the *first* twelve numbers of this type.

It is incorrectly stated, on page 26, that the first printed work to have its pages numbered with Arabic numerals was a certain work of Petrarch published in 1471. In the Annmary Brown Memorial of Providence, R. I., is a copy of a work by Werner Rolewinck published in 1470, and with pages so numbered (see *1921*, 423).

As a whole, this first volume of the new edition offers, at present, the most satisfactory treatment of its kind in the field discussed, and is a great improvement on the corresponding part of the first edition. There is no change in the general plan of the work. That the author has had the benefit of coöperative assistance on the part of G. Eneström and H. Wieleitner tends to fortify the feeling of the reader that down to the smallest detail great care has been exercised to make exact statements.

The Vereinigung wissenschaftlicher Verleger has, for some time, pursued a campaign of extortion so far as America is concerned, by charging for their books more than five times the published prices. For a bound copy of the book under review $2.30 are demanded. It is to be hoped that such action will be suitably resented.

August, 1921.

R. C. ARCHIBALD.

Technical and Scientific Serials in the Libraries of Providence, 1920. Compiled by the various libraries and edited by F. K. W. DRURY. Providence, R. I., 1921. Royal 8vo. 9 + 63 pp. Price 60 cents.

This publication lists about 2500 titles of serials (as defined by the American Library Association) in mathematics, chemistry, physics, biology, botany, and geology, as well as in the mechanic and useful arts, engineering, trades, medicine and the fine arts. "The list looks forward not simply to a complete list of *all* serials in the libraries of Providence or of the State, but to a complete sectional union list for New England and the East. Such a list is under way in the North Central states; this list is a contribution toward one on the Atlantic seaboard. With this in mind the national abbreviations for the libraries have been used rather than symbols purely local."

The publication is not above criticism in connection with the mathematical titles. A few selected indications of this must suffice. The two entries for this

MONTHLY on page 4 indicate confusion in the mind of the compiler and "Funkel" is given for "Finkel." On page 13, for "Férpussac" read "Ferussac." The entry "Notes in mathematics, 1905, 06, 12" under Johns Hopkins University is very misleading since these are only three of many such numbers in the *Johns Hopkins University Circulars* of which Brown University has practically a complete set. On page 19 it is stated that the first volume of the *Proceedings of the Edinburgh Mathematical Society* was published in 1883, when the correct date is 1894; volume two was published in 1884. *Revista Matematica hispano-americana* was not "continuing *Revista de matemáticas*, in Buenos Aires" (pages 49, 53).

In spite of its imperfections, however, the bibliographer and research student will be grateful for this valuable new aid in the promotion of their inquiries.

R. C. ARCHIBALD.

Mathematics of Finance. By H. L. RIETZ, A. R. CRATHORNE, and J. C. RIETZ. New York, Henry Holt and Company, 1921. 12mo. 13 + 280 pp. Price $3.00.

Preface: "It is the main purpose of this book to present a teachable elementary course in the application of mathematics to a broad class of financial problems.

"The material for the book has been obtained from many sources and tested in the teaching of such courses at three large universities. Experience has shown that the material is especially adapted to the needs of the students in schools and colleges of commerce and business administration, although general liberal arts and engineering students also find the course of much value.

"The man with a liberal business education should surely be thoroughly and accurately trained in the operation of interest in relation to finance. This course is designed to supply such training. In particular, this book treats of the relation of interest to the amortization of debts, to the creation of sinking funds, to the treatment of depreciation, to the valuation of bonds, to the accumulation of funds in building and loan associations, and to the elements of life insurance.

"Three chapters are devoted to an introduction to the elements of the mathematics of insurance. This is not a technical actuarial treatment of insurance, but simply a sufficient introduction to insurance so that the general business man who studies the book may obtain proper quantitive knowledge about the first principles of life insurance; and, as a student, may come to appreciate the beautiful system of long time finance involved in legal reserve life insurance.

"For the study of the book, no mathematical preparation, except that usually included in the high school course, is absolutely necessary, but courses in freshman and even in sophomore college mathematics will be found very useful, especially if only a short time is devoted to the work on this book.

"The plan of the book is such as to afford much elasticity in the time required to cover the work with a class. This is accomplished in part by the inclusion of many applied problems that should be solved by students when this work is given as a full course of, say three hours per week for a year. When the work is given as a briefer course, many problems may be omitted. Indeed, the entire work beginning with the chapter on probability, or with the previous chapter on building and loan associations may be omitted, without destroying the continuity of the course. These omissions would make possible a very brief course. Answers are given to some of the problems to meet the needs of those teachers who find them useful.

"The necessary subjects in pure mathematics beyond the elements of algebra are given in the body of the text, in foot-notes or in special chapters at the end of the text. These final chapters are on logarithms and progressions. Students who have not studied these subjects will do well to take portions of these chapters first.

"Some rather complicated formulas are developed in the book. It is important to guard the student against the tendency to substitute half blindly in the formulas. In fact, it is one of the features of our treatment to stress the formulas which the student should think out from first principles rather than the formulas most convenient for substitution without thinking.

"From experience in teaching the subject, we feel justified in the view that the careful study of the course presented in this book will do much to create in the business student an appreciation of exact science in business."

Contents—Chapter I: Interest, 1–30; II: Annuities certain, 31–61; III: The sinking fund method of paying a debt by periodical instalments, 62–76; IV: Valuation of bonds and other securities, 77–106; V: Mathematics of depreciation, 107–134; VI: The operation of funds in building and loan associations, 135–152; VII: Theory of probability with special reference to its application in insurance, 153–171; VIII: Life annuities, 172–189; IX: Net premium for some simple forms of life insurance, 190–202; X: Valuation of life insurance policies, 203–217; XI: Logarithms, 218–242; XII: Progressions, 243–248; Tables, 251–275; Index, 277–280.

NOTES.

The last number of *Proceedings of the Royal Society*, series A, volume 99, published September 1, 1921, contains biographical notices of five deceased members. These include sketches of Robert Bellamy Clifton (1836–1921; compare *1921*, 237) by R. T. G., vi–ix; of Srinivasa Iyengar Ramanuja Iyengar (1887–1920) by G. H. Hardy, xiii–xxix; and of Woldemar Voigt (1850–1919; compare *1921*, 32) by H. L., xxix–xxx. Hardy's sketch of Ramanujan appears to be identical with the one which he published earlier in *Proceedings of the London Mathematical Society* (see *1921*, 458).

Holland now publishes six mathematical journals, five of which are well known: *Nieuw Archief voor Wiskund*,[1] *Nieuw Tijdschrift voor Wiskunde*, *Revue Semestrielle des Publications Mathématiques*, *Wiskundige Opgaven met de Oplossingen*, and *Wiskundig Tijdschrift*. The sixth journal is a bimonthly periodical entitled *Christiaan Huygens, International Mathematisch Tijdschrift*, and the first number (64 pages) was published in October, 1921, by P. Noordhoff of Groningen who is also the publisher of the *Nieuw Tijdschrift*. It is planned that these periodicals shall be issued in alternate months (12 florins a year for the two; 8 florins for *Christiaan Huygens* alone). *Christiaan Huygens* is edited by Dr. F. Schuh, professor at the Technical University in Delft, with the collaboration of several others. The editors will accept articles (in English, Dutch, South-African, French, and German), which deal with topics of pure mathematics, mechanics, and mathematical physics. An honorarium of 20 florins per 16 pages is given for accepted articles. A few problems are also to be published in each number, the solutions appearing in subsequent issues. The standard of the publication is summed up in the announcement as follows: "In order to attract a wide field of readers and in that way to secure the largest useful result, the editors propose not to carry the contents to too high a level; it will have to remain largely within the field of study of doctors, doctorandi, engineers, and those who hold the diploma K_5." (The K_5-diploma is a diploma held by those who are entitled to teach in the highest type of secondary schools.)

Mathematical Philosophy, a Study of Fate and Freedom is the title of a new series of lectures by C. J. Keyser, crown 8vo, cloth, $4.20, published by E. P.

[1] This is the leading Dutch mathematical journal. It began as *Archief, uitgegeven door het Wiskundig Genootschap* in 1856; the third, and last, volume of this series was completed in 1875. The first volume of *Nieuw Archief voor Wiskunde* was started in 1875.

Dutton & Co., New York.—The Department of Commerce, Bureau of the Census, announces *United States Life Tables, 1890, 1901, 1910 and 1901–1910* [496 pp., 52 full page graphs, 22 diagrams, and 185 tables. Price $1.25].

The twenty-seventh volume of the *Bulletin of the American Mathematical Society* contains a portrait of Professor F. N. COLE and the following dedication:

By order of the Council of the Society, this volume is dedicated to

FRANK NELSON COLE

in appreciation of his devotion to the Society during his twenty-six years as Secretary and in recognition of his efficient leadership in the editorial work of the Bulletin for the past twenty-four years.

These are published with the concluding number (June–July, 1921).

ARTICLES IN CURRENT PERIODICALS.

ANNALES SCIENTIFIQUES DE L'ÉCOLE NORMALE SUPÉRIEURE (3d series), volume 38, January, 1921: "Aplatissement suivant l'axe polaire . . . d'une goutte liquide de révolution" . . . by J. Boussinesq, 1–12 [First paragraph: "Parmi les analogies physiques auxquelles pensèrent les théologiens du XIII^e siècle pour s'expliquer la sphéricité de la Terre, il y a celle de gouttes de pluie ou de rosée que l'on voit pendre aux feuilles des arbres, gouttes si bien arrondies surtout après s'être détachées pour tomber en chute libre. Ces théologiens sembleraient donc avoir admis, au moins implicitement, la fluidité primitive de notre globe, comme le firent d'une manière explicite, cinq cent ans plus tard, Newton et ses disciples en recourant à la pesanteur. Or il peut y avoir un certain intérêt théorique à poursuivre la même analogie des gouttes d'eau, mais d'une manière plus précise que ne l'a fait Plateau" . . .]—February to June: "Sur certaines fonctions automorphes de deux variables" by Georges Giraud, 43–164 [Quotation from page 55: "Voici donc le tableau d'ensemble de notre classification des substitutions linéaires: Substitutions hyperboliques; Substitutions elliptiques, à trois points doubles, à plan double—pénétrant a l'hypersphère principale, —extérieure a cette hypersphère; Substitutions paraboliques, à deux points doubles, à plan double, à point double unique; Substitution identique."]

THE ELECTRICIAN, volume 87, August 19, 1921: "The thermionic tube, a return to simplicity" by L. C. Pocock, 232–234 ["It is of course important to remember that the practical formulæ are only approximately true under finite conditions, and it is important to understand the differential equations associated with the action of the tube, but since the final formulæ in many cases depend upon the simplifying assumptions rather than on the differential equations, it may be of use to show how a simple explanation can be given"]—September 23: "Notes of the week: Progress in harmonic analysis," 373 ["Seeing however that in actual electromotive force waves, there are an infinite number of harmonics present and only a limited number of ordinates are drawn, though Mr. Clayton's method may work out well in practice, it is hardly likely to be held in very high favor among mathematicians who cling closely to Cambridge traditions"]; "The Heaviside unit and unit impulse functions" by A. Press, 376–377 ["Consider the functions, $y = x^m$, $U = x^{dm}$. We may define the Heaviside unity function, U, as the limiting form of the prior function as m approaches zero value from positive values of m only. However, the matter is not as simple as might appear due to the prevailing laxity in the use of the word 'limit' in mathematical literature. (Compare in this regard, *Fundamental Conceptions of Modern Mathematics* by Richardson and Landis, Open Court Publishing Company)" . . . "$x^0 = x^{dm} = U$. The index, zero, implies here that m approaches this value from plus values whence the index may be written dm if preferred."—A mathematical (?) exposition.]

L'ENSEIGNEMENT MATHÉMATIQUE, volume 21, 1920, nos. 5–6 (published July, 1921): "La notion d'équivalence dans la théorie des groupes" by G. A. Miller, 251–254; "Quelques

remarques sur un théorème relatif à la série hypergéométrique et sur la série de Kummer" by C. Cailler, 255–259; "Méthode à suivre pour déterminer la totalité des nombres premiers comprise entre 1 et le nombre $N = 1.2.3.5.7.11. \cdots p$, produit des $(\alpha+1)$ nombres premiers consécutifs pris a partir de 1" by E. Barbette, 260–265; "Sur l'équation fonctionelle $f[\varphi_1(t)] = f[\varphi_2(t)]$" by R. Wavre, 265–277; "Sur l'enseignement de calcul différentiel et intégral en Grèce" by P. Zervos, 278–281; "La préparation théorique et pratique des professeurs de mathématiques de l'enseignement secondaire en Argentine" by N. B. Moreno, 281–304; "La Commission Internationale de l'Enseignement Mathématique de 1908 à 1920.—Compte rendu sommaire suivi de la liste complète des travaux publiés par la Commission et les Sous-commissions nationales" by H. Fehr, 305–342; "Chronique," 343–349; "Bibliographie," 349–353; "Bulletin bibliographique," 354–366; Index, 367–370.

GIORNALE DI MATEMATICHE DI BATTAGLINI, volume 58, July–December, 1920: "Dimostrazione vettoriale di alcuni teoremi di Kasner," Nota di Clara Abbondati, 193–208 [Foot-note: "Edw. Kasner, 'The trajectories of Dynamics,' *Trans. American Mathem. Society*, v. 7, pp. 401–424, 1906; 'Dynamical trajectories: the motion of a particle in an arbitrary Field of Force,' *Ibid.*, v. 8, pp. 135–158, 1907."]

THE INTERNATIONAL STUDIO, volume 74, November, 1921: "Dynamic symmetry and its practical value today" by Maxwell Armfield, 76–85 [See *1920*, 314].

JOURNAL OF THE INDIAN MATHEMATICAL SOCIETY, volume 13, June, 1921: "Third conference of the Indian Mathematical Society," 81–89; "An extension of Feuerbach's theorem" by M. B. Rao, 109–116; Questions and Solutions, 117–120.

MATHEMATICAL GAZETTE, volume 10, October, 1921: "Mathematicians and their work" by L. J. Mordell, 321–323 [Review of F. Cajori, *A History of Mathematics*, 2d edition. "Prof. Cajori has taken the opportunity of adding a few new chapters, and of completely revising, in the light of modern knowledge, that part of his *History* dealing with the older work. The greatest attraction, however, of the volume is the account of the nineteenth and twentieth centuries, which now forms some two hundred of the present five hundred pages, making it almost a new history. There are two obvious difficulties in writing this section; not only are many of those mentioned still living, but of far more importance is the difficulty of keeping in touch with, and being conversant with, the relative importance of developments along so many different and unconnected lines. Prof. Cajori has solved the second difficulty by quoting extracts from reports and addresses by numerous experts. These, moreover, he has blended so well with his other material that he has produced a most useful and fascinating volume"]; "Mathematical Notes: A proof of the formula for the volume of a tetrahedron, in terms of the rectangular coördinates of its vertices" [not involving the square root which appears in most proofs but disappears in the result] by R. F. Muirhead, 324–325.

THE NATIONAL MARINE, New York, volume 18, July, 1921: "The evolution of the patent log. How early mariners measured distances through the water and the improvements that have been wrought" by Bradley Jones, 1–14.

NATURE, volume 108, October 6, 1921: "L'espace dans la chemie" (editorial), 171–172 [Review of D. A. Clibbens, *The Principles of the Phase Theory*. "The present work of Dr. Clibbens is the first book in English to attack seriously the real geometry of the subject. He carries us from the delightful simplicity of the binary systems right to the real thing—quaternary and quinary systems"]; Review of J. Cusack, *The Arithmetic of the Decimal System* by S. Brodetsky, 174–175 [The author does not adopt the metric system, but uses the decimal system with the present English units, the yard, the hundredweight, the gallon, the pound sterling, etc.]—October 13: "An algebraical identity" by R. F. Whitehead, 212 [Reply to an inquiry as to the primitive character of a solution of Gauss's cyclotomic equation]; "Notes," 222–226 ["In his address on October 3 to the newly constituted Glasgow Society for Psychical Research Sir Oliver Lodge gave an interesting exposition of his view of the æther of space as a region of possibilities in contrast with matter as a region of facts . . . he argued that if mind when dissociated from matter continues to exist, it can only be that there is something else which can perform the function of matter and serve as its instrument. For himself he has told us he is convinced that disembodied spirit personalities do exist in fact, and therefore for him it would seem the æther is a necessary postulate. His acceptance of the principle of relativity does not apparently in the least affect his belief in the real physical existence of the æther; it seems only to have added a few more negative qualities to that exceedingly elusive stuff and made its residual positive reality more than ever difficult to imagine. Still, perhaps the new society may succeed where Michelson and Morley failed, for psychical research, as Sir Oliver conceives it, is purely and essentially physical research, however

suspect to some of us its methods may appear."]—October 20: "The tendency of elongated bodies to set in the north and south direction" by A. Schuster, 240 [. . . "it may be pointed out that the earth's centrifugal force would act in a manner tending in the direction of the alleged effect, though the resulting couple is so minute that it would be extremely difficult to verify it experimentally. If a horizontal rod be placed in the north and south direction, its southern end is—in the northern hemisphere—further away from the earth's axis. The centrifugal force is therefore greater at the southern end, and if the rod be slightly displaced, the horizontal component of that force will tend to bring the rod back into the meridian plane."]; "Metaphysics and materialism" by H. W. Carr, 247–248 ["Materialism does not stand for any particular theory of the nature of matter, but for the general world-view that matter . . . exists and constitutes the reality of the universe, including reason and will, which as qualities or properties of some of its forms give rise to knowledge of it. This materialism reached the zenith of its expression in the Darwinian theory of natural selection, not in that theory itself . . . but in the implications which were generally accepted as contained in it, and especially in the application which was made of it to rationalize a world-view. It seemed to point a way by which it was possible to conceive, and to some extent to follow in its history, an evolution which had produced mind from an original matter. It may not be obvious at once that the mere rejection of the Newtonian concept of absolute space and time and the substitution of Einstein's space-time is the death-knell of materialism, but reflection will show that it must be so."]; "Notes" [Quotation from page 255: "*The Dictionary of Applied Physics* which Messrs. Macmillan and Co. Ltd. propose to issue under the editorship of Sir Richard Glazebrook is now in an advanced stage of preparation. The work will appear in five volumes of 600–700 pages each. . . . Dr. Horace Lamb has provided several articles on related mathematical questions."]; "Our astronomical column," 256 ["The Astronomical Union will meet next year in Rome, and among the Committee meetings that will be held there is one on calendar reform. . . . The main outlines of the reforms to be discussed include a more uniform arrangement of the lengths of the months, alteration in the position of the leap-day (the end of the year would be far more convenient from the point of view of astronomical tables), and making the incidence of the week-days the same every year by placing one day a year (two in leap-year) outside the weekly reckoning. . . . It is much easier to recognize the inconvenience of the present system than to agree on an alternative one."]

La Nature, volume 49_2, August 27, 1921: "Le problème de la suspension des voitures automobiles" by E. Weiss, 129–132—September 3: "Calculateur circulaire à disque mobile Arnault-Paineau" by J. Boyer, 70 supplément.

Periodico di Matematiche (See *1921*, 317) series 4, volume 1, no. 4, July, 1921: "Noterelle di logica matematica" by F. Enriques, 233–244; "I moti planetari e le leggi di Keplero" by E. Daniele, 245–262; "Gli inviluppi di linee curve ed i primordi del metodo inverso delle tangenti" by E. Bortolotti, 263–276; "Una interpretazione geometrica ed una estensione della divisibilità dei polinomî" by S. Pincherle, 276–282 [Suggests the representation of a polynomial of degree n by the point in space of $n + 1$ dimensions whose coördinates are the coefficients of the polynomial].

Philosophical Magazine, volume 42, September, 1921: "On certain fundamental principles of scientific inquiry" by Dorothy Wrinch and H. Jeffreys, 369–390; "The mental multiplication and division of large numbers" by V. A. Bailey, 390–397—October: "The convection coefficient in a dispersive medium" by A. Anderson, 509–511; "The forms of planetary orbits on the theory of relativity" by W. B. Morton, 511–522.

["Summary" (of article on certain fundamental principles of scientific inquiry): "It is shown that intensive theories of the structure of Nature which involve the use of infinite classes of entities of kinds known only by observation are not capable of yielding satisfactory accounts of the method by which we have acquired our knowledge of physics, and that the same applies to the theory of universal consent as a basis of scientific knowledge. The whole of a single person's knowledge is based upon a finite number of observations and his individual judgments, and the problem of a theory of scientific knowledge is to show how this can be carried out.

"It is clear that such a theory must depend on the theory of probability, and the question of the probability of physical laws and of inferences based on them is discussed. It is shown that it will never be possible to attach appreciable probability to an inference if it is assumed that all laws of an infinite class, such as all relations involving only analytic functions, are equally probable *a priori*. If inference is possible, the admissible laws must not be all equally probable *a priori*. It is suggested that all admissible laws can be arranged in a well-ordered sequence, each

having a finite prior probability, and such that each is more probable than any that follows it in the sequence. The probabilities of the laws must form a convergent series. On this basis it is shown that with sufficient empirical verification of a law the probability of further inferences from it will approach certainty.

"There is reason to believe that all admissible laws must form an enumerable aggregate, and this condition, and apparently all others which are necessary, are satisfied if we suppose that all laws admissible in physics are expressible as differential equations of finite order and degree, with rational coefficients. The most natural ways of well-ordering these are such that those of low order and degree, and involving no numerical constants other than small integers and fractions with small numerators and denominators, come earliest in the sequence. Accordingly, the practice of adopting the simplest law that fits the available observations appears to be closely related to the possibility of a satisfactory theory of inference."]

PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY, series 2, volume 20, part 3, August, 1921: "The classification of rational approximations" by P. J. Heawood, 233–240 [The construction of irrational numbers θ for which there are at most only a finite number of approximations x/y such that $|x/y - \theta| < 1/ky^2$ for a given value of k, x/y being a fraction in its lowest terms].

REVISTA DE MATEMATICAS Y FISICAS ELEMENTALES (See *1920*, 475; Spanish), volume 3, August, 1921: "Notes on analytical geometry" by E. Rebuelto, 73–78; "On projective coördinates" (conclusion) by F. D. Jaime, 78–84; "Note: On the sums of certain series" by J. S. Corti, 84–85—September: "The vectorial product" by T. Lalescu, 97–99; "Construction of conics" by J. S. Cortie, 99–101; "Notes: Rapid elimination of x from two equations of the third degree" by B. I. Baidaff, 106; "A relation of the theory of numbers" by T. I. G. Lintes, 106–107; Problems solved, Exercises, Problems proposed, 112–120.

REVUE GÉNÉRALE DES SCIENCES, volume 32, August, 1921: "Développement mathématique élémentaire de la relativité restreinte" by M. Sauger, 453–459 [A very simple derivation of the Lorentz equations and some mechanical and physical consequences]; Bibliographie: H. Broggi, *Análisis matemático*, 480 ["L'apparition de ce volume est un nouvel indice du mouvement qui tend à substituer, dans l'Amérique latine, aux ouvrages didactiques en langues étrangères ou aux traductions d'ouvrages étrangers, des œuvres dues aux professeurs et savants locaux."]—September: Bibliographie, 515, "Du Pasquier—le développement de la notion du nombre" [review] by Dr. Sorel.

REVUE SCIENTIFIQUE, volume 59, August 27, 1921: "La description géométrique des Alpes françaises" by L. R., 465 [Comments on the geodesic work of M. Helbronner as described in *La Montagne*, January, 1921]—September 10: "Notes et actualites: La Société des Amis des Sciences," 505–507 [Address of its President, M. Emile Picard. Quotation: "Mesdames, Messieurs, nous nous efforçons d'employer le mieux possible notre petite fortune, en suivant les intentions des fondateurs de la Société. Vous savez qu'ils s'étaient proposé de mettre à l'abri du besoin les savants et inventeurs peu fortunés, ainsi que leur famille après leur mort. Il nous était déjà impossible de remplir ce programme à la lettre avant la guerre, dans ces temps à la fois si proches et si lointains, où nous n'appréciions pas suffisamment, semble-t-il, la douceur de vivre à bon marché. . . . Quand ces temps heureux seront revenus, notre Société, recevant de toutes parts des dons généreux, deviendra une sorte de Ministère de la reconnaissance publique envers les savants et inventeurs malheureux. Je veux croire que ce n'est pas là un rêve; en attendant sa réalisation, continuons, mes chers collègues, notre propagande en faveur de l'œuvre qui nous est chère, et cherchons à lui amener de nouvelles recrues pour son action bienfaisante."]

SCHOOL SCIENCE AND MATHEMATICS, volume 21, October, 1921: "A dream come true" [an account of an experiment in teaching geometry inductively] by W. A. Austin, 621–627; "An extension of a process in factorization" by S. M. Karmalkar, 628–630; "Falling bodies in ancient and modern times" by F. Cajori, 639–648 [discussion of the question whether the story of Galileo's experiments at the tower of Pisa is mythical. See *Science*, n.s., volume 51, pp. 615–616, June 18, 1920; volume 52, pp. 272–273, September 17, 1920, and p. 409, October 29, 1920. "Recent criticism endeavors to sift the truth from a conglomerate mass of facts and fiction in the history of science. The movement is to be welcomed, whenever the work is carefully done. But an iconoclast may brush aside both facts and fiction, and leave unto the reader of science only a barren waste. The rebuilder of the historic structure may be an untrained architect with no sense of proportion or comprehension of the values of historic materials. The subject of falling bodies has had its trained historians and its amateurish historians. Comparatively simple questions have received different answers. Has Aristotle been misrepresented by Galileo to modern

writers? Did Aristotle actually claim that a heavier body falls faster in proportion to its weight? Some years ago an English writer, J. H. Hardcastle, denied this. Quoting a brief passage from Aristotle he argued that historians had grievously erred, that the great Peripatetic had been stupidly misunderstood. . . . Other of England's noted scientists made brief comments, basing their arguments on the one passage taken from Aristotle. Thus G. Greenhill, William Ramsay, and Oliver Lodge arrive at the conclusion that Aristotle has been misinterpreted, that he did not mean a body ten times heavier would fall from rest ten times faster, that he was thinking of a body moving through a resisting medium, like the air, which greatly modifies the motion, and was considering, not its initial, but its terminal velocity." They "did not themselves go back to search in Aristotle's writings; their deductions were based on the one quotation which had been brought to their attention. As a matter of fact, that one quotation gave only a partial exposition of Aristotle's theory."]; "Gerbert's letter to Adelbold" by G. A. Miller, 649–653; "The mathematics needed in freshman chemistry" by L. W. Williams, 654–665 [A classified list of over three hundred "mathematical" terms is given. The list includes such terms as "subdivisions," "double," "year," "1/3."]; "Do high school pupils dislike mathematics" by W. E. Gingery, 674–675; Problems and Solutions, 676–679; "Reception to the members of the National Committee of Mathematics Requirements for Secondary Schools" by W. J. Ryan, 696, 698—November: "An analysis of an experiment in teaching first year mathematics" by Ina E. Holroyd, 757–764; "Some plane geometry problems" by Lida C. Martin, 765–769; Problems and Solutions, 787–790; "The National Committee on Mathematical Requirements," 798, 800.

TÔHOKU MATHEMATICAL JOURNAL, volume 19, nos. 3–4, July, 1921: "The late Prof. Dr. Tetsugo Kojima" (Japanese), insert; "Sur les surfaces sphériformes" by G. Tiercy, 149–163; "Über Maxima und Minima von quadratischen Formen mit unendlichvielen Variabeln" by T. Kubota, 164–168; "Sur les équations différentielles homogènes" by F. Sibirani, 169–172; "On the roots of an algebraic equation" by M. Tajima, 173–174; "On the solutions of Mathieu's equations of the second kind" by S. C. Dhar, 175–182; "Cyclotomic sexe-section" by P. O. Upadhyaya, 183–186; "On the twisted cubic of constant curvature" by S. Narumi, 187–195; "On the characteristic property of the conic section" by S. Narumi, 196–204; "On the osculating conics of a plane curve" by T. Hayashi, 205–209; "Ueber gewisse Infinitesimal-Operationen der höheren Operationsstufen" by R. E. Moritz, 210–237; "Bemerkungen zu der Arbeit von Herrn Ogura: 'On the theory of the tides'" by O. Perron, 238–240; "Bemerkung über die Mittag-Lefflerschen Funktionen $E_\alpha(z)$" by G. Pólya, 241–248; "Pentaspherical geometries in noneuclidean space, I" by T. Ota, 249–270; Shorter notices and reviews, 271–275; Miscellaneous notes, 276–279.

ZEITSCHRIFT FÜR ANGEWANDTE MATHEMATIK UND MECHANIK, volume 1, August, 1921: "Über algebraische Gleichungen, die nur Wurzeln mit negativen Realteilen besitzen" by J. Schur, 307–311.

UNDERGRADUATE MATHEMATICS CLUBS.

All reports of club activities should be sent to E. L. DODD, 3012 West Ave., Austin, Texas.

CLUB ACTIVITIES.

THE MATHEMATICAL CLUB OF ADELPHI COLLEGE, Brooklyn, N. Y.

The Mathematical Club of Adelphi College was founded in the fall of 1898; and is apparently one of the oldest mathematical clubs in the country. Membership is open not only to students but to their friends. Meetings are held the second Wednesday of each month, and are well attended. The club has also a social meeting once a year.

During the year 1920–21 talks have been given upon the topics: "Lewis Carroll as a mathematician;" "Women as mathematicians;" "The trisection of an angle;" and certain topics suggested in THE AMERICAN MATHEMATICAL MONTHLY.

(Report by Evelyn Brisbane, secretary.)

THE MATHEMATICS CLUB OF COOPER UNION, New York City.

The Mathematics Club of Cooper Union was organized October 11, 1920. At this meeting it was decided to limit active membership for the current year to first-year and second-year

students. But on April 18, 1921, it was decided to open active membership for the next year to all classes.

Officers were elected January 24, and were re-elected April 18, as follows: Thomas Peterson '23, president; Karl Gerdin '24, secretary.

The students have given the Club their wholehearted support, and Night-School students also have been present. The attendance has varied from 15 to over 50.

The following programs were given:

October 11, 1920: Meeting of organization.
October 25: "Transfinite numbers" by Professor H. W. Reddick.
November 8: "Historical development of the valuation of π" by Reginald Overton '23.
November 22: "Mathematical fallacies" by Karl Gerdin '24.
December 6: "Magic squares" by Harry Serper '24.
December 20: "The trisection of the angle" by Peter Reiss '23.
January 10, 1921: "Non-Euclidean geometries" by Benjamin Levine '21.
January 24: Problem day. Also, election of officers.
February 7: "The fourth dimension" by J. J. Tanzola, instructor.
February 21: "Duality in mathematics" by David Samson '24.
March 7: "Finding the area of a parabola by the method of exhaustion" by Thomas Peterson '23.
March 28: "Numbers which form the sides of a right triangle" by Max Teitlebaum '24.
April 18: "The concept of infinity" by James Greene '21. Business meeting.

(Report by Mr. Gerdin.)

Denison Mathematics Club, Denison University, Granville, O.

[*1918*, 403; *1919*, 308, 362.]

The Denison Mathematics Club had a membership of forty-eight and an average attendance of twenty-two for the first semester of 1920–21. The plan of dividing the club into two groups for some of the meetings, for the freshmen and upper classmen respectively, is still on trial. The freshmen listen to talks on subjects which most of the upper classmen are acquainted with, and the upper classmen discuss topics which are rather advanced for the freshmen. The success of these group meetings is yet doubtful, but hopeful, since the average attendance of the freshmen this year is greater than last year. Problems for the two groups to solve between the times of meeting have stimulated interest.

Programs beginning with the fall of 1919 follow.

October 14, 1919: "Numbers" by Mr. R. A. Sheets, instructor.
October 28: "Arrangement of digits" by Mary Long '20; "Pascal triangle" by Marius Nielson '21; "Russian peasant's method of multiplication" by Esther Weaver '20; "Perfect numbers" by Professor F. B. Wiley.
November 25: "Outlines of some number systems" by Mr. Sheets; "The number system with 12 as a radix" by Richard Howe '20.
December 9: "The Chinese number system" by Hsieh Chang '22.
December 16: "A German astronomer's method of trisecting an angle" by Professor Wiley.
January 13, 1920: "Parallel coördinates" by Professor Wiley.
January 27: "Differential equations" by Professor Anna Peckham.
February 24: "Infinite roots of equations" by Ruby Robinson '21; "Complex numbers" by Burton Chandler '23.
March 23: "Mental tests" by Ruth Hendricks '23; "Mathematics used in the artillery" by Edward Boggs '20.
April 6: "Explanation of the adding machine" by Evangeline Nellis '22; "Use of the planimeter" by Richard Howe '20.
April 20: "Number pairs" by Esther Weaver '20; "Infinitesimals" by Marius Nielson '21.
May 4: "Space filling curves" by Albert Staniland '21; "Complex roots of unity" by William Newbury '23.
May 18: "Famous mathematicians" by Frances King '22; "Fourier series" by Bernard Lemon '22.
May 25: Annual banquet. Professor R. B. Allen of Kenyon College was the guest of honor.
October 5: "Magic Squares" by Mr. R. A. Sheets, instructor.
October 19: "Origin of number systems" by Professor Wiley.
November 2: "Development of number systems" by Grace E. Jefferson, instructor; Talk to freshmen by Professor Wiley.

November 16: "Simple harmonic motion" by Marius Nielson '21; Talk to freshmen by Mr. Sheets.
November 30: "Conic sections" by Grace McGrillis '23; "Numbers" by Mr. Sheets.
December 14: "Inversions" by Professor Peckham.
January 11, 1921: "Solutions of prize problems" by Marius Nielson '21 and Burton Chandler '23. Presentation of prizes.

(Report by Miss Hendricks.)

THE MATHEMATICS CLUB OF GOUCHER COLLEGE, Baltimore, Md.

[*1918*, 357; *1919*, 365; *1921*, 274.]

The programs for the first half of 1920–21 were as follows:
October 21, 1920: "Construction of the nine-point circle" by Eva Lazarus '22; "Proof that the nine-point circle is tangent to the inscribed and escribed circles" by Julia Sprenkel '22.
November 15: "Arithmetic calculation of cube roots and fifth roots" by Mildred Trueheart '22; "The slide rule" by Mary Lemon '22; "Proof of Feuerbach's theorem by methods of elementary analytics" by Professor Florence Lewis.
December 19: "History of three famous problems of construction" by Henrietta Morris '23; "Calculation of the index numbers of commodity prices" by Virginia Gallup '21.
January 13, 1921: "Flatland, by A. Square" by Gertrude Sandlass '23; "The history and significance of the concept of the fourth dimension" by Professor Clara L. Bacon.
February 17: "The theory of geometric inversion" by Margaret Sumwalt '23; "The theory and use of linkages in mechanical constructions" by Rose Diggs '22.

(Report by Professor Lewis.)

THE MATHEMATICS CLUB OF THE UNIVERSITY OF NORTH CAROLINA, Chapel Hill, N. C.

[*1918*, 90, 454.]

The officers elected for the year 1920–21 were: Professor Archibald Henderson, president; Professor A. W. Hobbs, vice-president; Professor J. W. Lasley, Jr., secretary.

The club numbers about fifty. The meetings take the form of mathematical smokers. Two prize problems presented by the Southern Engineering Corporation of Charlotte, N. C., added zest to the work of the club. The following programs have been given:
October 28, 1920: "The spirit of scientific research" by Professor William Cain; "Does minus three times minus three give plus nine?" by Professor Henderson.
November 18: "The contribution of Descartes" by Professor Lasley; "An interesting maximal case" by Herman Baisy, Gr.
December 9: "The geometric proposition and its converse" by Professor A. W. Hobbs; "A problem in envelopes" by Michael Hill, Gr.
January 20, 1921: "The Delian problem" by Professor A. S. Winsor; "Squaring the circle" by J. B. Linker, instructor; "Trisecting an arbitrary angle" by Walter Hook '23.
February 24: "The witch of Agnesi—an application" by Professor O. Stuhlman, of the department of physics; "The road bond issue mathematically treated" by Professor N. M. Paull.

(Report by Professor Lasley.)

THE MATHEMATICAL CLUB OF SMITH COLLEGE, Northampton, Mass.

[*1918*, 91, 455; *1920*, 184, 480.]

It has been the aim to make the meetings very informal, and to invite considerable discussion. Papers have been read during 1920–21 upon the following topics: "Lines of a triangle (altitudes, medians, bisectors)," "Area of a quadrilateral," "Nine point circle," "First work of mathematics printed in the New World," "Brocard points," "Simpson line of a triangle," "Conic sections as known by the Greeks," "Theory of quadratic equations."

The year closed with a social gathering. At the social meeting of the previous year, a play was given: "The Eternal Triangle." The prologue of this play was as follows:

"As is *congruent* with a meeting of the Mathematics Club of Smith College the program for this evening is a *problem* play. We have tried to *plot* in *graphic form* a *problem* which is by no means *imaginary*, but *real* and *common*; namely: 'The *Eternal Triangle*.' Given the three *ele-*

ments of the *triangle*, Mrs. *Adjacent Angle*, her *conjugate*, Mr. *Wright Angle*, and lastly Sir *Vertex*, an operator in home wrecking. In Mr. Wright Angle we find a *mixed expression* of love and business, business having the larger *proportion*. *Radically* speaking, Mrs. Adjacent Angle has an *inclination* towards loving love; and Sir Vertex fills in the *space* from which business has drawn her husband. But now—On with the *demonstration*." The tragedy closes thus: Adjacent Angle throws herself between the two. "Stop. *Project* your wrath no further. Don't go off on a *tangent*. It was not his fault. It was my *proposition* that he come. We have not lived on a *normal plane*. Your life is one *continuous function*. You scarcely ever go in my *circle*. You—do—not love me. I—" She turns to Sir Vertex who takes her in his arms. Wright Angle. "We have reached a *critical point*. The only *solution* is *division*. Go. Never come within my *radius* again." (Exit Sir Vertex. Adjacent Angle sinks to a chair and holds her head in her hand.)

(Report by Cassandana Page, secretary.)

THE PASCAL CIRCLE, TRINITY COLLEGE, Washington, D. C.

[*1920*, 425, 481; *1921*, 391.]

The following meetings were held—two in March and two in April—and at the close of the year there was a party at which refreshments were served and a puzzle was introduced with mathematical terms applicable to members of the Circle. At the regular meetings problems were frequently introduced and solved.

February: "The foundation of the Smithsonian Institute" by Margaret Crotty '21; "The first mathematical work published in America" by Clara Waldeck '23.

March: "Einstein and his theory" by Margaret Walsh '21; "A mathematical puzzle on the arrangement of coins, based on permutations" by Margaret Christie '23.

"Higher mathematics and chemistry" by Julia Thomas '22; "Two mathematical tombs" by Margaret Kelly '23.

April: "The correlation coefficient and the Rogers tests to determine mathematical ability" by Martha Crowly '22.

"A mathematician in love" (a poem) by Helen Ormand '23; "The place of mathematics in the curriculum" by Mary Brennan '21.

(Report by Miss Walsh.)

THE NEWTONIAN SOCIETY OF THE STATE COLLEGE OF WASHINGTON, Pullman, Wash.

[*1918*, 410.]

During the years 1919–20 and 1920–21 the following papers were read:

"Jordan curves" by Professor C. A. Isaacs; "Crap shooting" by Professor E. C. Colpitts; "Introduction to the fourth dimension" by Vera Roeder '20; "The wheel and shaftings of fourth-dimensional mechanics" by R. B. Kennedy, instructor; "Time as a fourth dimension" by F. N. Bryant, instructor; "History of π" by Corrine Barckly '21; "History of logarithms" by Florence Evans '21; "Perfect numbers" by Professor Colpitts; "War mathematics" by Robert Dixon '20; "Trisection of an angle" by Dorothea Sorenson '20; "One to one correspondence" by F. N. Bryant, instructor; "History of the Newtonian Society" by Eloise Brandt '22; "Area of a quadrilateral" by Alice Tardy '23; "Magic squares" by R. B. Kennedy, instructor; "Codes and ciphers" by Virginia Cooper '23; "What high school students like in mathematics" by Dorothea Sorenson '20; "One basis for business progress with a practical illustration" by Oliver E. Faulkner '21; "The theory of numbers" by Gladys Freeman, instructor; "Differential geometry" by Eloise Brandt '22; "Development of determinants" by Mildred Hunt '24; "Determinants in algebra" by Marian Ullery '24; "Determinants in geometry" by Twila Lewis '24; "The sixth dimension" by Professor Isaacs; "Non-Euclidean geometry" by Professor Colpitts; "Lemoine circles" by Alice Tardy '23; "Life of Newton" by Florence Evans '21; "Who is a mathematician?" by Professor Isaacs; "Limits in geometry" by Grace Barron '23; "Limits in calculus" by H. H. Irwin, instructor; "Probability" by Gwendolyn Thomas '21; "Probability" by R. B. Kennedy, instructor.

(Report by Miss Tardy, secretary.)

THE EUCLID CLUB OF THE UNIVERSITY OF WASHINGTON, Seattle, Wash.

[*1919*, 170.]

In November, 1920, the Mathematics Club of the University of Washington was reorganized under the name of the Euclid Club of the University of Washington, and the following officers were elected: Rubin Raport '23, president; Howard Robertson '23, vice-president; Lillie Siler '21, secretary.

The following programs were given:

December 16, 1920: "Einstein's theory of relativity" by Professor E. T. Bell.

January 20, 1921: "Greek numbers" by Helen Dunn '22.

February 3: "The Cyclo-harmonograph, a machine for drawing curves" by Professor R. E. Moritz (the inventor).

February 15: "Who's who in modern mathematics" by Gustene Rupe '23.

March 10: "Modern mathematical machines and famous mathematicians" by Lillie Siler '21.

April 7: "The use of mathematics in science" by Howard Robertson '23.

May 26: "The proof and use of the planimeter" by Rubin Raport '23.

(Report by Mr. Raport.)

NOTES.

Through the courtesy of one of our contributors, Mr. F. V. MORLEY, a Rhodes scholar from the United States at New College, Oxford, we are permitted to inspect the first two numbers of the manuscript *Proceedings of the Oxford University Undergraduate Mathematical Club*. The Club was started in October, 1920, and seven meetings were held during the year. One paper is read at each meeting and the *Proceedings* contain the complete papers. For the Michaelmas Term the papers were: "John Wallis" by J. S. HUGHES, of New College, and "Some circles connected with the triangle" by H. O. NEWBOLT, of Balliol. For the Hilary Term: "Introduction to inversive geometry" (2 parts) by F. V. MORLEY and "The twisted cubic" by Mr. TITCHMARCH, of Balliol. During the Summer Term two meetings were held in May, and Professor FRANK MORLEY, of Johns Hopkins University, and W. R. BURWELL, of Brown University, were the speakers.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. P. MANNING.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known text-books, or results found in readily accessible sources will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

2941. Proposed by W. D. CAIRNS, Oberlin College.

$1^2 + 2^2 + 3^2 + \cdots + (n-1)^2$ is a function of n. Find its derivative with respect to n.

2942. Proposed by L. E. DICKSON, University of Chicago.

I am dealt 13 cards at whist. What is the chance that all my cards will be diamonds?

2943. Proposed by L. E. DICKSON, University of Chicago.

In the game of bridge, what is my chance: (*a*) that my hand will contain 4 aces? (*b*) that some hand will contain 4 aces? (*c*) that my hand will be a Yarborough, *i.e.*, contain no honor?

2944. Proposed by S. A. COREY, Des Moines, Iowa.

A particle of mass m, starting from rest, is drawn by a string over a smooth horizontal plane, the other end of the string moving in the plane with uniform acceleration n along a line perpendicular to the initial position of the string. Prove that the tension of the string is $3mn \cos\theta$, where θ is the angle which the string makes with the given line. Also prove that the motion of the particle is vibratory.

2945. Proposed by T. M. BLAKSLEE, Ames, Iowa.

A point P in the plane of the triangle ABC rotates in a given direction around the vertices taken in either cyclical order, in each case through an angle equal to the corresponding angle of the triangle. That is, for example, AP rotates around A through an angle equal to the angle A of the triangle; then BP around B through an angle equal to the angle B, and so on. Prove that P coincides with its original position at the end of six of these rotations. (See problem 2899, *1921*, 228.)

2946. Proposed by H. C. BRADLEY, Massachusetts Institute of Technology.

Cut a regular hexagon into the smallest number of pieces that can be fitted together to form an equilateral triangle: (*a*) no piece to be turned over; (*b*) some pieces may be turned over.

2947. Proposed by D. H. MENZEL, Princeton University.

An oil tank has the shape of a cylinder with ends which are segments of a sphere and with horizontal axis. The diameter of the cylinder being given, and the radius of the spherical segments, derive a formula that will express the volume of the liquid contained in the tank in terms of its depth.

2948. Proposed by J. B. REYNOLDS, Lehigh University.

Find the envelope of the normal planes to the curve,

$$x = a\cos t, \qquad y = a(1 - \cos t), \qquad \text{and} \qquad z = a\sin t.$$

2949. Proposed by J. B. REYNOLDS, Lehigh University.

Find the lateral area of the cone with vertex at (0, 0, h) and whose base is the epicycloid,

$$x = \tfrac{3}{2}a\cos\theta - \tfrac{1}{2}a\cos 3\theta, \qquad y = \tfrac{3}{2}a\sin\theta - \tfrac{1}{2}a\sin 3\theta.$$

2950. Proposed by T. M. SIMPSON, JR., Randolph-Macon College, Ashland, Va.

Determine the curve which cuts the radius vector at an angle proportional to the radius vector.

NOTES.

25. The Area of a Quadrilateral.—The first expression for the area of an inscribed convex quadrilateral, in terms of its sides, was given by Brahmagupta (born 598 A.D.), without proof, in the following form:[1] "The product of half the sides and countersides is the gross area of a triangle and tetragone. Half the sum of the sides set down four times, and severally lessened by the sides, being multiplied together, the square root of the product is the exact area." The latter result appeared in a treatise written by Mahāvīrācārya, about 850 A.D., who gives[2] "The rule for arriving at the minutely accurate measurement of the

[1] *Algebra with Arithmetic and Mensuration from the Sanscrit of Brahmegupta and Bháscara.* Translated and edited by H. T. Colebrooke, London, 1817, pp. 295–296. See H. Weissenborn "Das Trapez bei Euklid" *Abhandlung zur Geschichte der Mathematik*, Heft 2, Suppl. historisch-literarischen Abtheilung, *Zeitschrift für Math. u. Physik*, vol. 4, 1879, pp. 181–184.

[2] *The Ganita-Sāra-Sangraha of Mahāvīrācārya* with English translations and notes. By M. Raṅgācārya, Madras, 1912, p. 198. Neither Brahmagupta nor Mahāvīrācārya knew that their rule for the exact area of a quadrilateral was only true for cyclic figures. The inexactness of the rule for all quadrilaterals was first pointed out by Bháskara (*l.c.*, § 167) who was born 1114 A.D.

area (of trilateral and quadrilateral figures):—. . . Four quantities represented (respectively) by half the sum of the sides as diminished by (each of) the sides (taken in order) are multiplied together; and the square root (of the product so obtained) gives the minutely accurate measure (of the area of the figure)." From this the area of a trilateral figure may be represented algebraically as[1] $\sqrt{[s(s-a)(s-b)(s-c)]}$, where s is half the sum of the sides whose lengths are a, b, and c; while the area of the quadrilateral figure[2] is

$$A = \sqrt{[(s-a)(s-b)(s-c)(s-d)]}, \tag{1}$$

where s is half the sum of the sides whose lengths are a, b, c, d. Moreover both Brahmagupta (*l.c.*, pp. 300–301) and Mahāvirācārya (*l.c.*, p. 199) state what is equivalent to the following expressions for the lengths of the diagonals of the quadrilateral:

$$m = \sqrt{\left[\frac{(ac+bd)(ab+cd)}{ad+bc}\right]}, \qquad n = \sqrt{\left[\frac{(ac+bd)(ad+bc)}{ab+cd}\right]}. \tag{2}$$

Formula (1) was rediscovered by Snellius who gives it in his commentary on the first book of Ludolphe van Ceulen's *De problematibus miscellaneis*.[3] The statement of the rule is given by Ozanam, without proof or reference to any other writer, in his *Cours de Mathematique* (Paris, vol. 3, 1690, p. 212; nouvelle édition, 1699, p. 147). Noticing this result Philip Naudé gave two demonstrations[4] of extraordinary complexity[5] to which Euler pays his respects introductory to furnishing a geometrical proof.[6]

Taking A, B, C, D as the vertices, in order, of an inscribed quadrilateral, and supposing AB, DC to meet in E, Euler first finds the following expression for the area, Q, of the quadrilateral,

$$QQ = \frac{1}{16}\cdot\frac{(AD-BC)(BE+CE+BC)}{BC}\cdot\frac{(AD-BC)(BE+CE-BC)}{BC}$$
$$\cdot\frac{(AD+BC)(BC+BE-CE)}{BC}\cdot\frac{(AD+BC)(BC-BE+CE)}{BC}.$$

He then shows that, apart from the numerical factor, the successive factors

[1] A result for which Heron of Alexandria gave an elegant proof hundreds of years earlier. The result was discovered by Archimedes who flourished still earlier.

[2] In connection with the following formulae of this note it will be supposed, unless otherwise stated, that a convex quadrilateral only is considered. The modifications for other cases are not difficult; some of them are later noted.

[3] *Oeuvres mathématiques de Ludolphe Van Ceulen*, traduites du hollandais en latin et enrichies de notes, par Snellius. Leyden, 1619. This was about 200 years before the first publication of Brahmagupta's results.—See Chasles, *Aperçu historique sur l'origine et le développement des méthodes en géométrie . . .* 2e édition, Paris, 1875, pp. 292, 432.

[4] "Demonstratio trium theorematum," *Miscellanea Berolinensia . . .* tome 3, 1727, pp. 259–269.

[5] C. L. A. Kunze comments on the proof which "zwar streng und bündig, aber mit einer *unerträglichen Weitschweifigkeit* abgefasst ist" (*Ueber einige theils bekannte, teils neue Sätze vom Dreieck und Viereck*, zweite vermehrte Ausgabe, Halle, 1848, p. 4).

[6] L. Euler, "Variae demonstrationes geometriae," *Nova acta acad. sc. Petrop.*, vol. 1 (1747–1748), 1750, pp. 57–63.

of the right-hand member are, respectively, $2S - 2BC$, $2S - 2DA$, $2S - 2AB$, $2S - 2CD$, where $2S = BC + DA + AB + CD$; whence he arrives at the form

$$Q = \sqrt{[(S - AB)(S - BC)(S - CD)(S - DA)]}.$$

A neat trigonometrical derivation of this result was given by Fuss[1] in a paper of considerable interest for the history of Poncelet polygons. Several other results here given will be noted in what follows.

The area was given in determinant form by Dostor:[2]

$$16A^2 = -\begin{vmatrix} -a & b & c & d \\ b & -a & d & c \\ c & d & -a & b \\ d & c & b & -a \end{vmatrix}.$$

Such a form would be suggested naturally to anyone familiar with formula (1), and noticing the factors of this determinant, with a substituted for $-a$, given by Ferrers[3] in 1861.

In 1782 Lhuilier gave,[4] in effect, the following expression for the radius, R, of the circle circumscribing the quadrilateral:

$$R = \frac{1}{4}\left[\frac{(ab + cd)(ac + bd)(ad + bc)}{(s - a)(s - b)(s - c)(s - d)}\right]^{\frac{1}{2}}. \tag{3}$$

For the numerator, within brackets, of the right-hand member, Klügel-Mollweide-Grunert substituted:[5] $abcd\Sigma a^2 + (abcd)^2\Sigma(1/a)^2$.

It was early recognized that, in general, there are three inscribed convex quadrilaterals with sides of given lengths; that these were equivalent in area but not superposable; that they have only three diagonals different in length. In 1626 Albert Girard stated[6] that the product of the lengths of the three different diagonals divided by twice the diameter of the circumcircle, is equal to the area of each of the quadrilaterals.

Hence, from (3), it appears that the product of the lengths of the three diagonals is $[(ab + cd)(ac + bd)(ad + bc)]^{1/2}$ which would be expected, since, from "Ptolemy's theorem," or (2), the product of the diagonals (m, n) of the quadrilateral whose sides are consecutively a, b, c, d, is given by $mn = ac + bd$.

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[1] N. Fuss, "De quadrilateris quibus circulum tam inscribere quam circumscribere licet," Conuentui exhib. die 14. Aug 1794, *Nova acta acad. sc. Petrop.*, vol. 10 (1792), 1797, pp. 103–125.

[2] G. Dostor, *Eléments de la théorie des déterminants*, Paris, 1877, p. 198.

[3] N. M. Ferrers, *An Elementary Treatise on Trilinear Coördinates*, Cambridge, 1861, p. 70.

[4] S. Lhuilier, *De Relatione Mutua . . . seu de Maximis et Minimis*, Warsaw, 1782, p. 21. Formula (3), in just this form, was given by Fuss (*l.c.*), p. 105.

[5] Klügel-Mollweide-Grunert, *Mathematisches Wörterbuch*, fünfter Theil, zweiter Band, Leipsic, 1831, p. 875.

[6] A. Girard, *Tables de sinus, tangentes et sécantes selon le raid de 10000 parties . . .*, 1626; Dutch translation, 1629. Compare Kästner, *Geschichte der Mathematik*, Göttingen, vol. 3, 1799, pp. 107–110; and S. Günther, *Vermischte Untersuchungen zur Geschichte der mathematischen Wissenschaften*, Leipsic, 1876, pp. 17–21.

The area, A', of the diagonal triangle of an inscribed quadrilateral was given by Dostor[1] in 1868 as

$$A' = \frac{4a^2b^2c^2d^2A}{(a^2b^2 - c^2d^2)(a^2d^2 - b^2c^2)};$$

whence Dostor found

$$\frac{A}{A'} = \frac{1}{4}\left(\frac{a^2}{c^2} - \frac{b^2}{d^2} + \frac{c^2}{a^2} - \frac{d^2}{b^2}\right) = \frac{1}{4}\left(\frac{ab}{cd} - \frac{cd}{ab}\right)\left(\frac{ad}{bc} - \frac{bc}{ad}\right).$$

With the above notation Dostor showed,[2] in 1848, that in the case of any quadrilateral,

$$16A^2 = 4m^2n^2 - (a^2 - b^2 + c^2 - d^2)^2. \tag{4}$$

He showed also that if the quadrilateral is circumscribed about a circle

$$\begin{aligned} A &= \tfrac{1}{2}\sqrt{[(mn + ac - bd)(mn - ac + bd)]}, \\ &= \tfrac{1}{2}\sqrt{[(mn + a'a'' - b'b'' + c'c'' - d'd'')(mn - a'a'' + b'b'' - c'c'' + d'd'')]}, \end{aligned} \tag{5}$$

where a', a''; b', b''; c', c''; d', d'' are the segments of the respective sides formed by the points of contact such that $a'' = b'$, $b'' = c'$, $c'' = d'$, $d'' = a'$.

Dostor found (*l.c.*, 1848, p. 73) that if p and q are the lengths of the line segments joining the middle points of pairs of opposite sides of any quadrilateral,

$$A = \tfrac{1}{2}\sqrt{[(p + q + m)(p + q - m)(p + q + n)(p + q - n)]}.$$

In 1874, he gave[3] yet another expression in terms of the coördinates of the four vertices:

$$2A = \begin{vmatrix} 1 & 0 & x_1 & y_1 \\ 0 & 1 & x_2 & y_2 \\ 1 & 0 & x_3 & y_3 \\ 0 & 1 & x_4 & y_4 \end{vmatrix}.$$

If a, b, c, d are the consecutive sides of a convex quadrilateral, and $\delta(\neq 0)$ is the length of the line joining the middle points of the principal diagonals, Catalan found[4] that

$$16\delta^2 A = (d^2 - b^2)\sqrt{[4a^2c^2 - (a^2 + c^2 - 4\delta^2)^2]} + (c^2 - a^2)\sqrt{[4b^2d^2 - (b^2 + d^2 - 4\delta^2)^2]}.$$

Strehlke showed[5] that if A and C are a pair of opposite angles of the quadrilateral

[1] G. Dostor, *Propriétés Nouvelles des Quadrilatères en Général* . . . Greifswald, [1868], p. 28; also in *Archiv der Mathematik und Physik*, vol. 48. In connection with this result E. W. Hobson erroneously omits the factor 4 (*A Treatise on Plane Trigonometry*, second edition, 1897, p. 205; fourth edition, 1918, p. 208).

[2] G. Dostor, *Nouvelles Annales de Mathématiques*, vol. 7, 1848, pp. 70 and 230; also vol. 33, 1874, p. 563.

[3] *Nouvelles Annales de Mathématiques*, vol. 33, 1874, p. 562; also *Archiv der Mathematik und Physik*, vol. 56, 1874, p. 240.

[4] E. Catalan, *Nouvelle Correspondance Mathématique*, vol. 6, 1880, pp. 52–53.

[5] Strehlke, "Zwei neue Sätze vom ebenen und sphärischen Viereck" . . ., *Archiv der Mathematik und Physik*, vol. 2, 1842, p. 324. J. F. König found (*Archiv* . . ., vol. 34, 1860, p. 14), for the area of a spherical quadrilateral the following expression which, in part, reminds one of Strehlke's formula:

$$A^2 = (s-a)(s-b)(s-c)(s-d) - abcd\cos^2\tfrac{1}{2}(A+C). \tag{6}$$

Hence[1] from formula (4)

$$\left.\begin{aligned} 16A^2 &= 4(ac+bd)^2 - (a^2-b^2+c^2-d^2)^2 - 16abcd\cos^2\tfrac{1}{2}(A+C), \\ &= 4(ac-bd)^2 - (a^2-b^2+c^2-d^2)^2 + 16abcd\sin^2\tfrac{1}{2}(A+C). \end{aligned}\right\} \tag{7}$$

If a circle can be inscribed in the quadrilateral, $a+c=b+d$, and this formula becomes[2]

$$A^2 = abcd\sin^2\tfrac{1}{2}(A+C).$$

If the quadrilateral is also inscribable[3] (Fuss, *l.c.*, p. 114),

$$A^2 = abcd; \tag{8}$$

under these conditions the radius of the inscribed circle is[4] $r = A/s = A/(a+c)$; for R. Klügel-Mollweide-Grunert wrote (*l.c.*)

$$R = \tfrac{1}{4}\sqrt{[\Sigma a^2 + abcd\Sigma(1/a)^2]}.$$

$$\sin\frac{E}{2} = \frac{\sqrt{[\sin(s-a)\sin(s-b)\sin(s-c)\sin(s-d) - \sin a\sin b\sin c\sin d\cos^2\tfrac{1}{2}(B+D)]}}{4\cos\tfrac{1}{2}a\cos\tfrac{1}{2}b\cos\tfrac{1}{2}c\cos\tfrac{1}{2}d}$$

$$+\frac{\sin a\sin b\sin B\cos\tfrac{1}{2}(c+d)\cos\tfrac{1}{2}(c-d) + \sin c\sin d\sin D\cos\tfrac{1}{2}(a+b)\cos\tfrac{1}{2}(a-b)}{8\cos\tfrac{1}{2}a\cos\tfrac{1}{2}b\cos\tfrac{1}{2}c\cos\tfrac{1}{2}d\cos^2\tfrac{1}{2}m},$$

where m is the diagonal AC, and E is the spherical excess of the quadrilateral $ABCD$. See also *Archiv. . . .*, vol. 4, 1843, p. 448.

[1] R. Baltzer, *Die Elemente der Mathematik*, Leipsic, vol. 2, 1862, p. 305.

[2] S. L. Loney, *Plane Trigonometry*, Cambridge, 1893, p. 253.

[3] Corresponding to formulæ (4), (5), and (8), P. Serret gave for the spherical quadrilateral, (*Des Méthodes en Géométrie*, Paris, 1855, p. 43):

$$(\sin\tfrac{1}{2}m\sin\tfrac{1}{2}n + \cos\tfrac{1}{2}a\cos\tfrac{1}{2}c - \cos\tfrac{1}{2}b\cos\tfrac{1}{2}d)$$

$$\sin^2\frac{E}{2} = \frac{(\sin\tfrac{1}{2}m\sin\tfrac{1}{2}n + \cos\tfrac{1}{2}b\cos\tfrac{1}{2}d - \cos\tfrac{1}{2}a\cos\tfrac{1}{2}c)}{4\cos\tfrac{1}{2}a\cos\tfrac{1}{2}b\cos\tfrac{1}{2}c\cos\tfrac{1}{2}d}; \tag{4'}$$

$$\sin^2\frac{E}{2} = \frac{\sin\tfrac{1}{2}(s-a)\sin\tfrac{1}{2}(s-b)\sin\tfrac{1}{2}(s-c)\sin\tfrac{1}{2}(s-d)}{\cos\tfrac{1}{2}a\cos\tfrac{1}{2}b\cos\tfrac{1}{2}c\cos\tfrac{1}{2}d}; \tag{5'}$$

$$\sin^2\frac{E}{2} = \tan\tfrac{1}{2}a\tan\tfrac{1}{2}b\tan\tfrac{1}{2}c\tan\tfrac{1}{2}d; \tag{8'}$$

where E is the spherical excess. From formula (5') we get (Grunert, *Nouvelles Annales de Mathématiques*, vol. 22, 1863, p. 336)

$$\cos^2\frac{E}{2} = \frac{\cos\tfrac{1}{2}s\cos\tfrac{1}{2}(s-a-b)\cos\tfrac{1}{2}(s-a-c)\cos\tfrac{1}{2}(s-a-d)}{\cos\tfrac{1}{2}a\cos\tfrac{1}{2}b\cos\tfrac{1}{2}c\cos\tfrac{1}{2}d}.$$

[4] Fuss found this result (*l.c.*, p. 112) by considering the question: What shall be the radius of a circle to which a quadrilateral with given sides is circumscribed, in order that the area of the quadrilateral shall be a maximum? He showed that it was when the quadrilateral was cyclic, *i.e.*, inscribed also in a circle. He found also an expression for the distance between the centers of these circles in terms of their radii. In this connection one may consult "The in-and-circumscribed quadrilateral" by W. E. Byerly (*Annals of Mathematics*, vol. 10, pp. 123–128, 1909); it is there shown that the necessary and sufficient condition for two circles (radii r, R) the distance between whose centers is x, to have a quadrilateral circumscribed to the first and inscribed to the second is that

$$\frac{1}{r^2} = \frac{1}{(x+R)^2} + \frac{1}{(x-R)^2}.$$

Déprez gave the following relation between the diagonals and the radii: $mn/2r = r + \sqrt{r^2+4R^2}$ (*Mathesis*, vol. 8, 1888, pp. 80, 104, 237–239). See also O. Kolberg, Program Braunsberg, 1853–1856.

It is well known that when there is one quadrilateral at the same time circumscribed about one circle and inscribed in another, there are an infinite number of quadrilaterals so related to these circles. The question naturally arises: Which one of these quadrilaterals has the greatest area? Or the least area? Welsch showed (*L'Intermédiaire des Mathématiciens*, vol. 16, 1909, p. 37) that the former occurred when the diagonals of the quadrilateral were at right angles to one another; that is, when two opposite vertices of the quadrilateral were at the extremities of the diameter of the circumcircle, center O, passing through the center, I, of the in-circle. He showed also that the quadrilateral is a minimum when it is an isosceles trapezium whose parallel sides touch the in-circle where it is intersected by OI.

Dostor showed [1] that in every quadrilateral:

(*a*) $$A=\tfrac{1}{4}(a^2-b^2+c^2-d^2)\cdot|\tan(m,n)|,$$

when (m, n) is not $90°$; [2]

(*b*) $$A=\tfrac{1}{4}(m^2-n^2)\cdot|\tan(p,q)|,$$

when (p, q) is not $90°$; and

(*c*) $$A=\tfrac{1}{2}mn\sin(m,n).$$

If g and h, k and l, are the segments of the interior diagonals,

$$A=\tfrac{1}{2}(gl+lh+hk+kg)\sin(m,n),$$

a form which Kummer found useful in his discussion of a rational quadrilateral, that is, one whose sides, diagonals and area are rational numbers [3] (*Journal für die reine und angewandte Mathematik*, vol. 37, 1848, p. 11).

If the exterior angles of an inscribed quadrilateral are bisected, the area of the quadrilateral formed by these bisecting lines is (E. W. Hobson, *Treatise on Plane Trigonometry*, second edition, Cambridge, 1897, p. 222)

$$\frac{1}{2}\frac{s^2(ab+cd)(ad+bc)}{(a+c)(b+d)\sqrt{[(s-a)(s-b)(s-c)(s-d)]}}.$$

Meier Hirsch gave the following formula (*Sammlung geometrischer Aufgaben*, Erster Teil, Berlin, 1805, p. 36) for the area of a quadrilateral whose sides are known and in which a pair of opposite angles, (a, d) and (b, c) are equal:

$$A=\frac{1}{4}\frac{ad+bc}{ad-bc}\sqrt{[(a+b+c+d)(a+b-c-d)(a+d-b-c)(b+d-a-c)]}.$$

When the opposite angles are right angles P. F. Verhulst showed that we have

[1] G. Dostor, *Propriétés Nouvelles des Quadrilatères en Général . . .*, Greifswald, [1868], pp. 3, 6, 7.

[2] If the quadrilateral is circumscribable about a circle this formula becomes

$$A=\tfrac{1}{2}(bd-ac)\cdot\tan(m,n)$$

(C. Davison, *Subjects for Mathematical Essays*, London, 1915, p. 33).

[3] The problem of the rational quadrilateral has an extensive history; see the discussion by L. E. Dickson in this MONTHLY, *1921*, 244–250; and in his *History of the Theory of Numbers*, vol. 2, 1920, pp. 216–221.

(*Correspondance Mathématique et Physique* (Quetelet), Brussels, vol. 6, 1830, p. 121) $A = (s-a)(s-d) = (ad+bc)/2$.

Other expressions for the area are given in E. Heis and T. J. Eschweiler, *Lehrbuch der Geometrie, Dritter Teil, Ebene und sphärische Trigonometrie*, zweite Auflage, Köln, 1875, pp. 86–90, and in M. Hirsch, *l.c.*, pp. 33–41.

In 1782 Lhuilier considered the question of when the polygon with given sides should have a maximum area, and found that this occurred when it was inscribed in a circle.[1] The result for the quadrilateral follows at once on setting $A + C = 180°$, $\cos C = -\cos A$, in formula (6) or the first part of formula (7). Lhuilier considered also the question of when the area is a minimum and showed[2] that this arose when the quadrilateral was no longer convex but when the sides cut one another. From the second part of formula (7) it appears that the area is a minimum when $A + C = 0$, that is, when[3] $\cos C = \cos A$, the case of the inscribed concave quadrilateral. In this case

$$16A_1^2 = 4(ac-bd)^2 - (a^2-b^2+c^2-d^2)^2,$$

$$= (a+b+c+d)(-a-b+c+d)(-a+b-c+d)(-a+b+c-d),$$

$$= -\begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix}, \tag{9}$$

which is Ferrers's determinant referred to above. From this, it follows at once that for any convex quadrilateral

$$\sqrt{[s(s-a-d)(s-b-d)(s-c-d)]} \gtrdot A,$$

a result given by Wolstenholme[4] in one of his problems. Among a number of other maxima and minima results connected with the quadrilateral, R. Sturm proved[5] that for given sides, in a given order, the inscribed convex quadrilateral has the greatest diagonal product and hence the greatest area (compare (4)).

In *Astronomische Nachrichten*, no. 42, November, 1823, Shumacher reports that on page 61 of the description by Möbius of the observatory at Leipzig the following problem occurs: Let *ABCDE* be any five points in the plane joined so as to form the five triangles *ABC, BCD, CDE, DEA, EAB* whose areas α, β, γ, δ, ϵ, respectively, are known; required the area of the pentagon *ABCDE*.

[1] S. Lhuilier, *l.c.*, p. 5; see also Enneper "Ueber das Maximum eines Vierecks von gegebenen Seiten," *Gött. Nachrichten*, 1885, pp. 175–180. The same result holds true for a spherical quadrilateral (W. J. McClelland and T. Preston, *Treatise on Spherical Trigonometry*, part 2, London, 1903, p. 50.)

[2] S. Lhuilier, *l.c.*, pp. 23–24; formula (9) was given, in effect, by Lhuilier. Details of this question are discussed by R. Sturm and E. Lampe in *Journal für die reine und angewandte Mathematik*, vol. 96, 1884, pp. 78–80.

[3] R. Baltzer, *Die Elemente der Mathematik*, vol. 1, 1860, pp. 129–130; vol. 2, 1862, p. 306. For inscribed quadrilaterals $A^2 - A_1^2 = abcd$ (Möbius, *Journal für die reine und angewandte Mathematik*, vol. 3, 1828, pp. 17–18).

[4] J. Wolstenholme, *A Book of Mathematical Problems*, London, 1867, problem 377; third edition, 1878, problem 596. It is clear that Wolstenholme might have written "$< A$" instead of "$\gtrdot A$."

[5] R. Sturm, *Maxima and Minima in der elementaren Geometrie*, Leipsic, 1910, p. 25.

Shumacher appends a solution handed to him by Gauss (*Carl Friedrich Gauss Werke*, vol. 4, 2ter Abdruck, 1880, pp. 406–407); this leads to the equation

$$A^2 - (\alpha + \beta + \gamma + \delta + \epsilon)A + (\alpha\beta + \beta\gamma + \gamma\delta + \delta\epsilon + \epsilon\alpha) = 0.$$

P. Serret noted (*Nouvelles Annales de Mathématiques*, vol. 7, 1848, p. 28) that if the pentagon becomes the quadrilateral $ABCD$, α, β, γ being the areas of the triangles ABC, BCD, CDA, the above equation for the area of the quadrilateral becomes $A^2 - (\alpha + \beta + \gamma)A + \alpha\beta + \beta\gamma = 0$,—which can, on solving, be at once verified.

The problem of the construction with ruler and compasses of an inscribed quadrilateral, being given its sides, has an interesting history extending over five hundred years. This is, in general, only a particular case of the problem discussed by Ozanam (*Dictionaire Mathematique*, Amsterdam, 1691, pages 461–464): "Construct the quadrilateral, of given area, given the lengths of its four sides." R. C. ARCHIBALD.

SOLUTIONS

2832 [1920, 227]. Proposed by S. A. COREY, Des Moines, Iowa.

Prove that the square of the sum of four squares is the sum of four squares, that the square of the sum of eight squares is the sum of six squares, and that the square of the sum of sixteen squares is the sum of ten squares.

I. SOLUTION BY THE PROPOSER.

Given the identity,

$$\begin{aligned}
&(Pp+Pq+Qp-Qq-Rr+Rs-Sr-Ss-Tt-Tu+Ut-Uu-Vv-Vx+Xv-Xx)^2\\
&\quad+(Pr+Ps-Qr+Qs-Rp+Rq+Sp+Sq-Tv+Tx+Uv+Ux+Vt-Vu-Xt-Xu)^2\\
&\quad+(Pp-Pq-Qp-Qq+Rr+Rs-Sr+Ss+Tt-Tu+Ut+Uu+Vv-Vx+Xv+Xx)^2\\
&\quad+(Pr-Ps+Qr+Qs+Rp+Rq+Sp-Sq+Tv+Tx+Uv-Ux-Vt-Vu-Xt+Xu)^2\\
&\quad+(Pt+Pu+Qt-Qu-Rv-Rx+Sv-Sx+Tp+Tq-Up+Uq+Vr-Vs+Xr+Xs)^2\\
&\quad+(Pv-Px-Qv-Qx-Rt+Ru-St-Su+Tr+Ts-Ur+Us-Vp+Vq-Xp-Xq)^2\\
&\quad+(Pt-Pu-Qt-Qu+Rv-Rx+Sv+Sx-Tp+Tq-Up-Uq-Vr-Vs+Xr-Xs)^2\\
&\quad+(Pv+Px+Qv-Qx+Rt+Ru-St+Su-Tr+Ts-Ur-Us+Vp+Vq-Xp+Xq)^2\\
&\qquad=2(P^2+Q^2+R^2+S^2+T^2+U^2+V^2+X^2)(p^2+q^2+r^2+s^2+t^2+u^2+v^2+x^2). \quad (1)
\end{aligned}$$

Add $(P^2+Q^2+R^2+S^2+T^2+U^2+V^2+X^2)^2$ and $(p^2+q^2+r^2+s^2+t^2+u^2+v^2+x^2)^2$ to each member of (1). The second member then becomes the square of the sum of 16 squares, and the first member becomes the sum of 10 squares, as required. If $P = Q = R = S = t = u = v = x =$ zero, it follows that the square of the sum of eight squares is the sum of six squares. That the square of the sum of four squares is the sum of four squares is a direct consequence of Euler's well known theorem: (sum of four squares) (sum of four squares) = (sum of four squares).

II. Mr. NORMAN ANNING, of the University of Michigan, contributes formulæ exhibiting these squares as the sum of 3, 5 and 9 squares, instead of 4, 6, and 10. The results are given for record.

Corresponding to $(x_1^2 + x_2^2)^2 = (x_1^2 - x_2^2)^2 + 4x_1^2x_2^2$, we have

$$(x_1^2 + x_2^2 + x_3^2 + x_4^2)^2 = (x_1^2 - x_2^2 + x_3^2 - x_4^2)^2 + 4(x_1^2 + x_3^2)(x_2^2 + x_4^2)$$

= the sum of three squares.

$$(x_1^2 + x_2^2 + \cdots + x_8^2)^2$$

$$= (x_1^2 - x_2^2 + - + \cdots - x_8^2)^2 + 4(x_1^2 + x_3^2 + x_5^2 + x_7^2)(x_2^2 + x_4^2 + x_6^2 + x_8^2)$$

= the sum of five squares, since by Euler's Identity the last term can be expressed as the sum of four squares.

Similarly,

$$(x_1^2 + x_2^2 + \cdots + x_{16}^2)^2$$

$$= (x_1^2 - x_2^2 + - + \cdots - x_{16}^2)^2 + 4(x_1^2 + x_3^2 + \cdots + x_{15}^2)(x_2^2 + x_4^2 + \cdots + x_{16}^2)$$

= the sum of nine squares.

The fact that the product of two numbers each of which is the sum of eight squares is itself the sum of eight squares is stated, with a reference to "Thomson, 1877," on page 115 of Carmichael's *Diophantine Analysis*, 1915.

An identity for the eight-square case is as follows:

$$\begin{aligned}(a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + g^2 + h^2) \times (p^2 + q^2 + r^2 + s^2 + t^2 + u^2 + v^2 + w^2)\\ \equiv (ap + bq + cr + ds + et + fu + gv + hw)^2 + (aq - bp + cv - du + ew + fs - gr - ht)^2\\ + (ar - bv - cp - dt + es - fw + gq + hu)^2 + (as + bu + ct - dp - er - fq - gw + hv)^2\\ + (at - bw - cs + dr - ep + fv - gu + hq)^2 + (au - bs + cw + dq - ev - fp + gt - hr)^2\\ + (av + br - cq + dw + eu - ft - gp - hs)^2 + (aw + bt - cu - dv - eq + fr + gs - hp)^2.\end{aligned}$$

NOTE (written after the above was in type)—Euler's identity given in a letter to Goldbach dated May 4, 1748 (*Correspondance Mathématique et Physique* ed. by Fuss, vol. 1, 1843, p. 452) was as follows: "Si $m = aa + bb + cc + dd$ et $n = pp + qq + rr + ss$ erit $mn = A^2 + B^2 + C^2 + D^2$ existente $A = ap + bq + cr + ds$, $B = aq - bp - cs + dr$, $C = ar + bs - cp - dq$, $D = as - br + cq - dp$." This was generalized by C. F. Degen in 1822 (*Mém. Acad. Sc. St. Pétersbourg*, vol. 8 [1817–18], pp. 207–219) as follows:

$$\begin{aligned}(a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + g^2 + h^2) \times (p^2 + q^2 + r^2 + s^2 + t^2 + u^2 + v^2 + w^2)\\ \equiv (ap + bq + cr + ds + et + fu + gv + hw)^2 + (aq - bp + cs - dr + eu - ft + gw - hv)^2\\ + (ar - bs - cp + dq \mp ev \pm fw \pm gt \mp hu)^2 + (as + br - cq - dp \pm ew \pm fv \mp gu \mp ht)^2\\ + (at - bu \pm cv \mp dw - ep + fq \mp gr \pm hs)^2 + (au + bt \mp cw \mp dv - eq - fp \pm gs \pm hr)^2\\ + (av - bw \mp ct \pm du \pm er \mp fs - gp + hq)^2 + (aw + bv \pm cu \pm dt \mp es \mp fr - gq - hp)^2.\end{aligned}$$

Cf. L. E. Dickson, "On quaternions and their generalization and the history of the eight square theorem," *Annals of Mathematics*, second series, volume 20, p. 164, 1919.

2840 [1920, 274]. Proposed by NORMAN ANNING, University of Michigan.

It is observed in a table of values of

$$\log_{10}(\operatorname{colog}_{10} x)$$

that second differences are zero for values of x in the neighborhood of 0.37. Prove that this must be the case. (Cf. Chappell's *Five-Figure Mathematical Tables*, Edinburgh, 1915, p. 180.)

SOLUTION BY C. C. WYLIE, University of Illinois.

Let $y = \log_{10}(\operatorname{colog}_{10} x)$ and $M = \log_{10} e$. Then

$$y = M \log\left(\log\frac{1}{x}\right) + M \log M.$$

$$\frac{dy}{dx} = -\frac{M}{x \log\frac{1}{x}}, \qquad \frac{d^2y}{dx^2} = \frac{M\left(\log\frac{1}{x} - 1\right)}{\left(x \log\frac{1}{x}\right)^2} = 0,$$

if $x = 1/e = 0.368$ approx. Therefore, the second differences in a table of values of this function must be practically zero for values of x near 0.368.[1]

Also solved by H. L. OLSON, H. S. UHLER, A. H. WILSON, and the Proposer.

2842 [1920, 274]. Proposed by H. S. UHLER, Yale University.

Express explicitly the following sextic in x as the product of a quadratic and a biquadratic:

$$3x^6 - 6k_1x^5 + (7k_1^2 - 9k_2^2)x^4 - 2(2k_1^3 - 4k_1k_2^2 - 3k_3^3)x^3 + [(k_1^2 - k_2^2)^2 - 9k_1k_3^3]x^2 - (k_1^2 - 2k_2^2)(k_1k_2^2 - 9k_3^3)x + (k_1^2 - 3k_2^2)(k_2^4 - 3k_1k_3^3).$$

I. SOLUTION BY W. D. CAIRNS, Oberlin College.

If $k_1 = k_2 = 0$, the expression can be factored thus:

$$(3x^4 + k_3^4)(x^2 - 3k_3^2).$$

If $k_1 = k_2 = 0$, it can be factored thus:

$$3x^3(x^3 + 2k_3^3) \text{ or } (3x^4 + 6k_3^3x)x^2.$$

If $k_2 = k_3 = 0$, it can be factored thus:

$$x^2(3x^3 - 3k_1x + k_1^2)(x^2 - k_1x + k_1^2).$$

These will be consistent only if one factor contains, at least, the terms $3x^4 - 3k_1x^3 + k_1^2x^2 + 6k_3^3x + k_3^4$, and the second $x^2 - k_1x + k_1^2 - 3k_3^2$. The last term of the given expression shows that the first factor must contain also the term $-3k_1k_3^3$, and multiplication of these two tentative factors shows that the term $-k_1k_2^2x$ completes the first factor. Thus we have as the two factors: $3x^4 - 3k_1x^3 + k_1^2x^2 - (k_1k_2^2 - 6k_3^3)x + (k_3^4 - 3k_1k_3^3)$ and $x^2 - k_1x + (k_1^2 - 3k_2^2)$.

II. REMARKS BY H. P. MANNING, Providence, R. I.

The polynomial can be written $u_3k_3^3 + v_6$, where u_3 and v_6 are polynomials of degrees 3 and 6 in x, k_1 and k_2. If there are rational factors k_3^3 must go entirely with one factor, since u_3 and v_6 are not the cubes of rational expressions. The factor which does not contain k_3 will be the highest common factor of u_3 and v_6. We can also factor u_3, noting that it contains only the second power of k_3.

2845 [1920, 326]. Proposed by E. L. POST, Princeton University.

Prove that if y_x is a solution of the functional equation

$$y_x = \frac{y^2_{x+1}}{x} + y_{x+1}$$

for positive integral values of x with $y_x > 0$, then

$$\lim_{x \to \infty} y_x \log x = 1.$$

SOLUTION BY THE PROPOSER, AND OTTO DUNKEL, Washington University.

If x and y_x are both positive, the equation

$$y_x = \frac{y^2_{x+1}}{x} + y_{x+1} \tag{1}$$

has one positive root,

$$y_{x+1} = \frac{\sqrt{x^2 + 4xy_x} - x}{2}.$$

From (1), we clearly have then that

$$y_x > y_{x+1} > 0. \tag{2}$$

[1] Since the second difference $y(a + 2h) - 2y(a + h) + y(a) = h^2y''(\theta)$, where $a < \theta < a + 2h$.—EDITORS.

The equation may be written

$$\frac{1}{y_{x+1}} - \frac{1}{y_x} = \frac{y_{x+1}}{xy_x}.$$

Hence

$$\frac{1}{y_x} - \frac{1}{y_1} = \sum_1^{x-1} \frac{1}{x} \cdot \frac{y_{x+1}}{y_x} = \sum_1^{x-1} \frac{1}{x}\left[1 - \frac{y_x - y_{x+1}}{y_x}\right] = \sum_1^{x-1} \frac{1}{x} - \sum_1^{x-1} \frac{1}{x^2} \cdot \frac{y^2_{x+1}}{y_x}.$$

Now

$$\sum_1^{x-1} \frac{1}{x} = \log x + r,$$

where r approaches Euler's constant C when x becomes infinite (Goursat-Hedrick, vol. 1, p. 103, Ex. 1); so that, if we substitute and divide by $\log x$, we have

$$\frac{1}{y_x \log x} - 1 = \frac{\dfrac{1}{y_1} + r - \displaystyle\sum_1^{x-1} \frac{1}{x^2} \cdot \frac{y^2_{x+1}}{y_x}}{\log x}.$$

But from (2) $y_{x+1} < y_x < y_1$, $y^2_{x+1} < y_1 y_x$. Therefore

$$\sum_1^{x-1} \frac{1}{x^2} \cdot \frac{y^2_{x+1}}{y_x} < y_1 \sum_1^{x-1} \frac{1}{x^2} < \frac{\pi^2 y_1}{6}.$$

The numerator of the fraction remains numerically less than a fixed number when x becomes infinite and we have

$$\lim y_x \log x = 1.$$

It is to be noticed that the above method can be immediately extended to

$$y_x = f(x) y^2_{x+1} + y_{x+1},$$

for certain functions $f(x)$, where $\sum_1^\infty [f(x)]^2$ is a convergent series.

NOTE. From (1) we obtain

$$\frac{y_x}{y_{x+1}} - 1 = \frac{y_{x+1}}{x}, \quad \text{and} \quad x\left(\frac{1}{y_{x+1}} - \frac{1}{y_x}\right) = \frac{y_{x+1}}{y_x}. \tag{3}$$

When x becomes infinite, the first equation in (3) together with (2) shows that y_x/y_{x+1} approaches unity. Hence the second equation gives

$$\lim_{x \to \infty} x\left(\frac{1}{y_{x+1}} - \frac{1}{y_x}\right) = 1.$$

It is easily seen that the limit is the same if we replace x by $x + 1$ and when this is done it follows that

$$\lim_{x \to \infty} \frac{1}{\log (x+1) y_{x+1}} = 1,$$

by use of the theorem on page 108, § 162, E. Cesàro, *Elementares Lehrbuch der algebraischen Analysis* . . ., Leipzig, 1904. The desired result easily follows from the above.

2846 [1920, 326].

Find the entire volume within the surface $x^{1/2} + y^{1/2} + z^{1/2} = a^{1/2}$. (W. A. Granville, *Elements of Differential and Integral Calculus*, revised ed., 1911, p. 420.)

This equation, rationalized, is the equation of Steiner's quartic surface, every tangent plane to which cuts it in two conics. (Cf. Salmon-Rogers, *Analytic Geometry of Three Dimensions*, 5th ed., vol. 2, 1915, pp. 171, 201, 207, 213f. Also C. M. Jessop, *Quartic Surfaces* 1916, chapter 7.)

I. SOLUTION BY L. A. EASTBURN, North Arizona Normal School, Flagstaff, Ariz.

The required volume inclosed by the surface is

$$v = \int_0^a \int_0^{(a^{1/2} - x^{1/2})^2} \int_0^{a^{1/2} - x^{1/2} - y^{1/2})^2} dz dy dx,$$

$$= \int_0^a \int_0^{(a^{1/2}-x^{1/2})^2} [(a^{1/2} - x^{1/2})^2 - 2(a^{1/2} - x^{1/2})y^{1/2} + y]dydx,$$

$$= 1/6 \int_0^a [a^2 - 4a^{3/2}x^{1/2} + 6ax - 4a^{1/2}x^{3/2} + x^2]dx,$$

$$= 1/6 \left[a^3 - 8/3a^3 + 3a^3 - 8/5a^3 + \frac{a^3}{3}\right] = \frac{a^3}{90}.$$

II. Notes by R. C. Archibald, Brown University.

The integral here arising is a particular case of one considered by Lejeune Dirichlet in 1839.[1] If $V = \int x^{a-1}y^{b-1}z^{c-1} \cdots dxdydz \cdots$, considering all positive values of $x, y, z, \cdots$ such that

$$\left(\frac{x}{\alpha}\right)^p + \left(\frac{y}{\beta}\right)^q + \left(\frac{z}{\gamma}\right)^r + \cdots < 1$$

the constants $a, b, c, \cdots, p, q, r, \cdots, \alpha, \beta, \gamma, \cdots$ being also positive, then

$$V = \frac{\alpha^a\beta^b\gamma^c \cdots}{pqr \cdots} \cdot \frac{\Gamma\left(\frac{a}{p}\right)\Gamma\left(\frac{b}{q}\right)\Gamma\left(\frac{c}{r}\right)\cdots}{\Gamma\left(1 + \frac{a}{p} + \frac{b}{q} + \frac{c}{r} + \cdots\right)}.$$

For the surface $(x/a)^{1/n} + (y/b)^{1/n} + (z/c)^{1/n} = 1$, the volume in the first octant would be $abc(n!)^3/(3n)!$ When $n = 7/2$ we have a result given[2] in 1883

$$V = \frac{abc}{(2/7)^3} \cdot \frac{[\Gamma(7/2)]^3}{\Gamma(23/2)}.$$

The values, for $n = 3/2$ and $n = 1/2$ were given in Williamson, *Elementary Treatise on the Integral Calculus*, 6 ed., 1891, pp. 289–290. The result for the surface $(x/a)^2 + (y/b)^2 + (z/c)^4 = 1$, given in Todhunter, *A Treatise on the Integral Calculus*, 4 ed., 1874, p. 186, follows at once from the Dirichlet formula above.

2852 [1920, 377]. Proposed by D. H. Richert, Bethel College, Newton, Kans.

What is the radius of a cylinder inscribed in a right cone, radius of base $R = 5$ inches, and altitude $h = 18$ inches, the volume of the cylinder to be $1/n$ $(= 3/4)$ that of the cone?

Solution by H. S. Uhler, Yale University.

Let V denote the volume of the cone, and let r, v, and z denote, respectively, the radius, the volume, and the altitude of the cylinder. Then

$$V = \tfrac{1}{3}\pi hR^2, \qquad v = \pi zr^2, \qquad \text{and} \qquad v = \frac{1}{n}V;$$

hence,

$$zr^2 = \frac{hR^2}{3n}. \tag{1}$$

From the similar right triangles obtained by passing a plane through the common axis of the cone and cylinder we find

$$z = \frac{h(R - r)}{R}. \tag{2}$$

These two equations are homogeneous in h and z, and also in R and r; therefore they determine the ratio of the altitudes and the ratio of the radii independently as functions of n alone.

Substituting from (2) for z in equation (1) we obtain

$$r^3 - Rr^2 + R^3/(3n) = 0.$$

[1] *Comptes Rendus . . . de l'Académie des Sciences*, vol. 8, p. 156; also in *Journal de Mathématiques Pures et Appliquées*, vol. 4, p. 168.

[2] *Mathematical Questions with their solutions from the "Educational Times,"* vol. 38, p. 104.

Let $\rho \equiv r/R$ so that the cubic becomes

$$\rho^3 - \rho^2 + 1/(3n) = 0. \tag{3}$$

Since the constant term of this equation is positive there is always one real negative root, and this is inadmissible. Therefore the remaining roots must be real in order that a positive root may exist. The derivative of the left-hand member of (3) is $\rho(3\rho - 2)$ which vanishes for $\rho = 0$ and $\rho = 2/3$. The expression for the derivative shows that there is a maximum for $\rho = 0$, and a minimum for $\rho = 2/3$; the value of the minimum is $1/3(1/n - 4/9)$ and hence if $1/n > 4/9$ there is only one real root, the negative root already mentioned.

Consequently the given value $1/n = 3/4$ conflicts with the preceding condition and makes the problem impossible.

Also solved by T. M. BLAKSLEE, L. A. EASTBURN, R. A. JOHNSON, R. H. MARSHALL, J. Q. MCNATT, ARTHUR PELLETIER, and A. V. RICHARDSON.

2855 [1920, 377]. Proposed by J. L. RILEY, Stephenville, Texas.

Show that the circle of curvature at any point of the ellipse cannot pass through the centre unless the eccentricity be greater than $1/\sqrt{2}$.

I. SOLUTION BY A. V. RICHARDSON, Bishop's College, Lennoxville, Quebec.

In the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a > b),$$

the normal at the point $(a \cos\theta, b \sin\theta)$ is

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2. \tag{1}$$

To find the coördinates of the center of curvature, we solve (1) and its derivative with respect to θ,

$$\frac{ax \sin\theta}{\cos^2\theta} + \frac{by \cos\theta}{\sin^2\theta} = 0,$$

giving

$$x = \frac{a^2 - b^2}{a} \cos^3\theta; \qquad y = -\frac{a^2 - b^2}{b} \sin^3\theta. \tag{2}$$

The (radius of curvature)^2 is, of course, the (distance)^2 between this point and $(a \cos\theta, b \sin\theta)$ *i.e.*,

$$\frac{(a^2 \sin^2\theta + b^2 \cos^2\theta)^3}{a^2 b^2}.$$

If we put the radius of curvature equal to the distance of the center of curvature from the origin we shall get after some reductions

$$(a^2 - 2b^2) \cos^2\theta = (2a^2 - b^2) \sin^2\theta. \tag{3}$$

Hence, for real values of θ, we must have $a^2 > 2b^2$, *i.e.*,

$$a^2 > 2a^2 (1 - e^2) \qquad \text{or} \qquad e > \frac{1}{\sqrt{2}}.$$

II. REMARKS BY OTTO DUNKEL, Washington University.

If we determine the condition that the perpendicular bisector of the segment from the origin to the point $(a \cos\theta, b \sin\theta)$ shall pass through the center of curvature we shall get equation (3) more easily.

Or we may express this by saying that the center of curvature projects into the middle point of the segment. This condition takes the form $2(x_1 x_c + y_1 y_c) - (x_1^2 + y_1^2) = 0$, which reduces to the same equation (3).

We find also that e may equal $1/\sqrt{2}$. In the ellipse for which $e = 1/\sqrt{2}$ the radius of curvature at the extremity of the major axis is equal to $a/2$.

Also solved by A. M. HARDING, WILLIAM HOOVER, and H. S. UHLER.

NOTES AND NEWS.

It is to be hoped that readers of the MONTHLY will coöperate in contributing to the general interest of this department by sending items to H. P. MANNING, Brown University, Providence, R. I.

Mr. J. W. BLINCOE has been appointed instructor of mathematics in Randolph-Macon College, Ashland, Va.

Mr. F. J. BURKETT, of the Pennsylvania State College, was appointed instructor of mathematics in the University of Pittsburgh in September, 1920.

At Columbia University, the following are temporarily associated with the department of mathematics as visiting lecturers: Professor M. A. NORDGAARD, of the University of Iowa, Professor J. A. NORTHCOTT, of Syracuse University, and Professor R. R. HITCHCOCK, of the University of North Dakota. Professor H. B. MITCHELL is absent on leave for the first half of the current academic year, and Professors D. E. SMITH and EDWARD KASNER will be absent during the second half of the year.

Mr. A. O. HICKSON, of Acadia University, has been appointed instructor of mathematics at Brown University for the second semester of the present academic year, Professor R. C. ARCHIBALD being absent on sabbatic leave (cf. *1921*, 336).

Mr. C. J. CARDIN, mechanical engineer and industrial instructor of the Bethlehem Shipbuilding Company, has been appointed instructor of Mathematics at Washington and Jefferson College.

Professor T. O. WALTON, of Michigan Agricultural College, has been appointed assistant professor of mathematics at Kalamazoo College.

Dr. L. H. HILL, of the University of Montana, has been appointed associate professor of mathematics at the University of Maine.

W. H. HILL, who during the past year was instructor of mathematics at the University of Colorado, is now assistant professor of mathematics at the Manual Training Normal School at Pittsburg, Kansas.

Mr. F. W. JOHN, of Cornell University, has been appointed instructor of mathematics in Washington Square College of New York University.

Mr. A. S. PRATT, who was instructor of mathematics at the University of Maine 1919–1921 (*1920*, 141), is teacher of mathematics at the Moses Brown School, Providence, R. I.

Miss HELEN BARTON, for the past two years instructor of mathematics at Wellesley College (*1919*, 219), has been appointed dean of women at Albion College, Albion, Mich.

Dr. H. R. KINGSTON, formerly assistant professor of mathematics and astronomy, University of Manitoba, has been appointed professor and head of the department of mathematics at Western University, London, Ontario.

Last June Lecturer JOSEPH DE PEROTT, of Clark University, retired from active service in the department of mathematics.

Instructor H. L. SMITH, of the University of Wisconsin, and a charter member of the Mathematical Association, has been appointed professor of mathematics at the University of the Philippines.

At Dartmouth College, Assistant Professor R. D. BEETLE has been promoted to a full professorship. Professor J. W. YOUNG has returned to the department after a leave of absence of two years during which he has served as chairman of the National Committee on Mathematical Requirements. Assistant Professor F. M. MORGAN has been granted leave of absence for the second semester of the current academic year.

At Cornell University, Professor JAMES MCMAHON retired from active service last June. The previously announced Heckscher research grant to Mr. H. S. VANDIVER (*1921*, 287) for the continuation of his investigations on the theory of algebraic numbers has been extended to cover the first semester of the year 1921–1922, and Mr. VANDIVER has been granted leave of absence for that period. Mr. JESSE OSBORNE, of Pennsylvania State College, and Mr. R. L. JEFFERY, of Acadia College, have been appointed instructors.

Mr. G. I. HOPKINS, head of the mathematics department and for the last three years principal of the Manchester, N. H., high school, resigned from this position in June. He had reached the age of 72 and had given faithful service to this school for a continuous period of forty-one years. He is the author of *Manual of Plane Geometry on the Heuristic Plan*, Boston, 1891.

Miss HENRIETTA S. LEAVITT, member of the staff of the Harvard Astronomical Observatory, died in Cambridge, Mass., December 12, 1921. She was born in Lancaster, Mass., July 4, 1868. She graduated at Radcliffe College with the degree of B.A. in 1892, took graduate work in astronomy in 1892–1893, and research work at the Harvard Observatory in 1895–1896. In 1902 she became engaged in photographic researches, which she continued till her death, making a special study of the photographic brightness of the stars and of the relation between the brightness and periods of variable stars. The scientific results of her work are found in volumes 60, 71, 84 and 85 of the *Annals of the Harvard Observatory*.

The following reports of Summer Sessions in 1922 have been received.

University of California, June 26–August 5. In addition to the usual courses in Plane trigonometry, Graphic algebra, Differential calculus, Integral calculus, and Plane analytic geometry, the following courses are offered. By Professor G. E. F. SHERWOOD: Solid analytic geometry and theory of infinite series. By Dr. P. H. DAUS: Synthetic projective geometry. By Professor B. A. BERNSTEIN: Elementary algebra for advanced students. By Professor L. E. DICKSON: Applied trigonometry, Selected topics in the theory of equations. By Professor G. D. BIRKHOFF: Differential equations, Theory of relativity. Each course is equivalent to two units.—Southern Branch, July 1–August 12. In addition to courses in Plane trigonometry, Plane analytic geometry, and Differential calculus, the following course is offered. By Professor D. N. LEHMER: Introduction to synthetic projective geometry, 2 units.

University of Chicago, First Term, June 19–July 26; Second Term, July 27–September 1. By Professor E. H. MOORE: Fundamental number systems of

analysis, four hours, first term only; Limits and series, four hours, first term only. By Professor H. E. SLAUGHT: Elliptic integrals, four hours; Differential equations, four hours. By Professor G. A. BLISS: Functions of a complex variable, four hours; Calculus of variations, four hours. By Professor E. J. WILCZYNSKI: Metric differential geometry, four hours; College algebra, five hours. By Professor J. W. A. YOUNG: Theory of equations, four hours; Differential calculus, five hours. By Professor F. R. MOULTON: Celestial mechanics, four hours. By Professor A. C. LUNN: Statistics and probability, four hours; Vector analysis, five hours. By Professor R. L. MOORE: Foundations of analysis situs, four hours; Integral calculus, five hours. By Mr. J. H. VANDIVER: Theory of numbers, four hours; Plane analytic geometry, five hours. By Professor C. H. GINGRICH: Plane trigonometry, five hours.

Columbia University, July 10–August 18. In addition to the usual courses in Logarithms and trigonometry, Solid geometry, College algebra, Analytic geometry, and Calculus, the following courses are offered. By Professor J. F. RITT: Theory of functions of a complex variable. By Professor DUNHAM JACKSON: Differential equations, Fourier's series. By Professor W. B. FITE: Projective geometry. By Dr. G. A. PFEIFFER: Introduction to higher algebra. Each of these courses meets five times per week.

Cornell University, July 8–August 18. In addition to the courses in Solid geometry, Advanced algebra, Trigonometry, Analytic geometry, and Calculus, the following advanced courses are offered. By Professor VIRGIL SNYDER: Projective geometry. By Professor D. C. GILLESPIE: Analysis. The following reading and research courses are also offered. By Professor SNYDER: Algebraic curves and surfaces. By Professor F. R. SHARPE: Applied mathematics. By Professor W. B. CARVER and Professor F. W. OWENS: Foundations of geometry and problems in synthetic geometry. By Professor GILLESPIE: Functions of a real variable, Point-sets, Calculus of variations. By Professor W. A. HURWITZ: Advanced analysis. By Professor C. F. CRAIG: Functions of a complex variable.

Harvard University, July 10–August 19. By Mr. R. E. LANGER: Trigonometry, Analytic geometry. By Professor W. C. GRAUSTEIN: Differential and integral calculus. By Professor O. D. KELLOGG: Differential and integral calculus. Each course meets five times per week.

University of Illinois, June 19–August 12. In addition to the usual courses in College algebra, Plane trigonometry, Analytic geometry, Differential and integral calculus, the following courses are offered. Each course meets five times per week, except as otherwise stated. By Professor G. A. MILLER: Advanced algebra; Critical study of Cajori's *History of Mathematics*, 3 hours; Seminar in group theory, 2 hours. By Dr. C. C. CAMP: Differential equations. By Dr. C. F. GREEN: Advanced analytic geometry. By Professor E. B. LYTLE: Teachers' course. By Professor ARNOLD EMCH: Geometric transformations. By Professor J. B. SHAW: Vector methods.

University of Iowa, First Term, June 7–July 22. By Professor J. F. REILLY: Subject matter and teaching of mathematics, Actuarial theory. By Professor

E. W. CHITTENDEN: Integral calculus, Differential equations, Theory of functions of a complex variable. By Mr. R. E. GLEASON: Differential calculus, Theory of equations, Projective geometry. By Mr. F. S. HARPER: Algebra, Trigonometry. By Mr. O. E. BROWN: Trigonometry, Analytic geometry. By Mr. H. W. CHANDLER: Solid geometry. Second Term, July 24–August 25. By Professor R. P. BAKER: Calculus, Projective geometry, Geometric magnitudes. By Dr. W. H. WILSON: Algebra, Determinants. By Mr. R. E. GLEASON: Analytic geometry. The courses are five hours per week in class through the sessions. When transformed into hours of the ordinary academic year, the credit for each course taken during the first session is two semester hours and for the second session five sixths of this amount.

University of Kansas, First Session, June 12–July 21. By Professor C. H. ASHTON: Complex numbers, College algebra. By Professor U. G. MITCHELL: Differential equations, Teachers' course. By Professor J. J. WHEELER: Analytic geometry, Calculus. Second Session, July 24–August 18. By Professor S. LEFSCHETZ: Theory of equations, Trigonometry.

University of Michigan, June 26–August 18. By Professor W. B. FORD: Advanced algebra, Advanced calculus. By Professor Peter FIELD: Theory of potential. By Professor T. R. RUNNING: Graphical methods. By Professor L. C. KARPINSKI: History of mathematics. By Professor T. H. HILDEBRANDT: Differential equations. By Professor W. B. CARVER: Theory of statistics, Theory of probability. By Mr. N. H. ANNING: Advanced geometry. By Mr. C. H. RICHARDSON: Finite differences. Each course meets for four hours a week.

University of Minnesota, June 19–July 29. In addition to the usual courses in College algebra, Plane trigonometry, Analytic geometry, and Differential and integral calculus, the following course is offered: By Professor R. W. BRINK, W. L. HART, and A. L. UNDERHILL: Selected topics in advanced mathematics.

University of Oklahoma, June 7–August 1. In addition to courses in College algebra, Trigonometry, Analytic geometry, Differential calculus, and Integral calculus, the following courses are offered. By Professor S. W. REAVES: Analytic geometry of space, 3 hours. By Professor J. O. HASSLER: Teachers' course, 2 hours. By Professor N. ALTSHILLER-COURT: History of mathematics, 2 hours; College geometry, 3 hours.

University of Wisconsin, June 19–August 4. In addition to courses in Solid geometry, College algebra, Trigonometry, Analytic geometry, Differential and integral calculus, and Mathematical theory of investment, the following courses are given. By Professor E. B. SKINNER: Elements of the group theory. By Professor H. W. MARCH: Mathematical formulation of scientific problems, Fourier series. By Professor ARNOLD DRESDEN: Advanced integral calculus, Theory of point sets. By Professor W. W. HART: The teaching of mathematics, Higher Euclidean geometry for teachers. By Professor E. P. LANE: Differential geometry. By Dr. F. E. ALLEN: Theory of equations. By Mr. H. T. DAVIS: Differential equations.

National Committee on Mathematical Requirements.

The complete report of the National Committee on Mathematical Requirements is in the press and will, it is hoped, be ready for distribution in April. [1] It is published under the title "The Reorganization of Mathematics in Secondary Education" and will constitute a volume of about 500 pages. The table of contents given below indicates its general character.

Through the generosity of the General Education Board the National Committee is in a position to distribute large numbers of this report free of charge. It is hoped that the funds available will be sufficient to place a copy of this report in every regularly maintained high school library, and also to furnish every individual with a copy free of charge who is sufficiently interested to ask for it. Requests from individuals for this report are now being received. They should be sent as early as possible to J. W. Young, Chairman, Hanover, New Hampshire. The receipt of these requests is not in general acknowledged, but applicants may rest assured that their requests will be filled when the report is ready for distribution. If the number of requests received exceeds the number the Committee is able to distribute, the earlier requests will receive the preference.

The table of contents of the report is as follows: Part I: General Principles and Recommendations—Chapter I: A brief outline of the report; II: Aims of mathematical instruction, general principles; III: Mathematics for years seven, eight and nine; IV: Mathematics for years ten, eleven and twelve; V: College entrance requirements; VI: Lists of propositions in plane and solid geometry; VII: The function concept in secondary school mathematics; VIII: Terms and symbols in elementary mathematics. Part II: Investigations Conducted for the Committee—Chapter IX: The present status of disciplinary values in education, by Vevia Blair; X: The theory of correlation applied to school grades, by A. R. Crathorne; XI: Mathematical curricula in foreign countries, by J. C. Brown; XII: Experimental courses in mathematics, by Raleigh Schorling; XIII: Standardized tests in mathematics for secondary schools, by C. B. Upton; XIV: The training of teachers of mathematics, by R. C. Archibald; XV: Certain questionnaire investigations; XVI: Bibliography on the teaching of mathematics, by D. E. Smith and J. A. Foberg.

[1] The Committee has since announced that the report will not be ready before May.

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THE PATH OF LIGHT IN A GRAVITATIONAL FIELD.

By H. S. UHLER, Yale University.

While reading the recent books on the generalized principle of relativity, Einstein's theory, etc., I failed to find any numerical data which would give, even approximately, a quantitative idea of the angular deviation at various points on the curved ray which represents the course taken by a light wave in coming from a remote star to the earth after having passed close to the limb of the sun. All of the books examined give the total deviation,[1] $4m/R = 1.745''$, from infinity to infinity (that is, the angle between the asymptotes to the ray) but nothing more about numerical angles. Since my investigation of this problem may contain something of interest to the readers of the MONTHLY, it seems appropriate to present the details of the analysis in this place.

As a purely mathematical problem, we shall first find the polar equation of a ray of light in an unlimited medium in which the index of refraction is a linear function of the reciprocal of the distance of the wave from the pole or center of symmetry. We shall then simplify the equation by adapting it to the special case of a gravitational field. Finally, we shall obtain so much of a series expansion as will be necessary for the calculation of the very small angular deviations involved.

In order to avoid undue distraction in the midst of the subsequent analysis we shall now state a theorem,[2] borrowed from geometrical optics, which will constitute the keynote to the plan of attack to be followed in this paper. For a transparent medium having the property that the index of refraction possesses spherical symmetry with respect to a given center, the product of the index of refraction at any point on a ray of light in the medium by the length of the perpendicular dropped from the center of symmetry upon the tangent to the ray at the point in question is constant. Let $n_1, n_2, n_3, \cdots$ denote the indices of refraction at points along the same ray and let $p_1, p_2, p_3, \cdots$ symbolize the lengths of the corresponding perpendiculars, then

$$n_1p_1 = n_2p_2 = n_3p_3 = \cdots = constant. \tag{1}$$

This fact is independent of the functional relation between the index of refraction and the distance from the center of symmetry.

In the figure let S denote the center of symmetry and let ABC represent the curved ray of light. B is the point on the ray nearest to S. Take the line SB as the axis of polar coördinates (r, θ) of any point P on the ray. $SP = r$, $\angle BSP = \theta$. Draw tangents BT and FP to the curved ray at the points B

[1] R denotes the radius of the sun and m a quantity which Eddington calls "the mass of the sun in astronomical units," while Weyl styles it the "gravitational radius." m is always expressed in terms of a unit of length, usually the kilometer.

[2] *Cf.* R. S. Heath, *Geometrical Optics*, 1887, p. 328.